1. Let $\vec{r}(t) = \left\langle \frac{5}{t}, \sqrt{t-2}, \frac{t^2-9}{t-3} \right\rangle$.

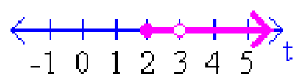
   (a) Find $\lim\limits_{t \to 3} \vec{r}\,(t)$.

$$\begin{aligned}
\lim_{t \to 3} \vec{r}\,(t) &= \lim_{t \to 3} \left\langle \frac{5}{t}, \sqrt{t-2}, \frac{t^2-9}{t-3} \right\rangle \\
&= \left\langle \lim_{t \to 3} \frac{5}{t}, \lim_{t \to 3} \sqrt{t-2}, \lim_{t \to 3} \frac{t^2-9}{t-3} \right\rangle \\
&= \left\langle \frac{5}{3}, 1, \lim_{t \to 3} \frac{2t}{1} \right\rangle \\
&= \left\langle \frac{5}{3}, 1, 6 \right\rangle
\end{aligned}$$

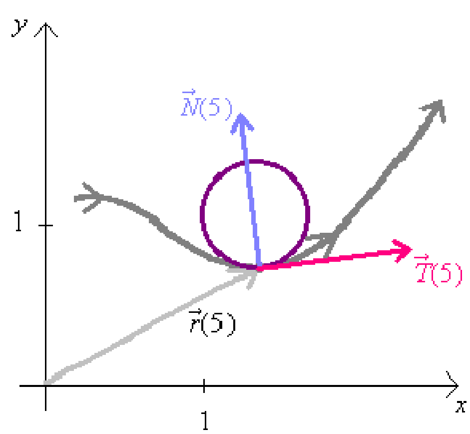
   (b) Find the domain of $\vec{r}\,(t)$.

$$\begin{aligned}
\text{For the 1st component, we need } t &\neq 0. \\
\text{For the second component, we need } t &\geq 2. \\
\text{For the third component, we need } t &\neq 3.
\end{aligned}$$

Thus, the domain of $\vec{r}\,(t)$ is $[2, 3) \cup (3, \infty)$.

On a number line: 

2. Let $\vec{r}\,(t)$ describe the 2-D curve sketched below. The arrows on the curve indicate the direction the curve is traced as t increases.



Add each of the following to the sketch:

   i. $\vec{T}\,(5)$,

   ii. $\vec{N}\,(5)$, and

   iii. The osculating circle at the point where $t = 5$.

3. Consider the curve defined by $\vec{r}(t) = \langle 2t^2, t, t^2 \rangle$

(a) This curve intersects the plane $x + y - 2z = 1$ at a single point. Find the coordinates of the point of intersection.

$$\begin{aligned} 2t^2 + t - 2t^2 &= 1 \\ t &= 1 \end{aligned}$$

$$(x, y, z) = \left(2(1)^2, 1, 1^2\right) = (2, 1, 1)$$

(b) Find the angle between the curve and the plane $x + y + 2z = 4$ at the point of intersection you found in part (a). Your answer may include the inverse cosine function.

$$\begin{aligned} \vec{r}\,'(t) &= \langle 4t, 1, 2t \rangle \\ \vec{r}\,'(1) &= \langle 4, 1, 2 \rangle \quad \text{(This is the direction vector of the curve at the point of intersection.)} \end{aligned}$$

$\vec{n} = \langle 1, 1, 2 \rangle$ is the normal vector for the plane.

The angle α between these two vectors is :

$$\begin{aligned} \cos\alpha &= \frac{\vec{r}\,'(1) \cdot \vec{n}}{|\vec{r}\,'(1)|\,|\vec{n}|} = \frac{\langle 4, 1, 2 \rangle \cdot \langle 1, 1, 2 \rangle}{\sqrt{4^2 + 1^2 + 2^2}\sqrt{1^2 + 1^2 + 2^2}} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}} \\ \alpha &= \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \end{aligned}$$

The angle between the plane and the curve is $\dfrac{\pi}{2} - \alpha = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{3}{\sqrt{14}}\right)$

4. Let $\vec{r}(t) = \langle t, 2\cos t, 2\sin t \rangle$.

(a) Find $\vec{N}\left(\frac{\pi}{3}\right)$.

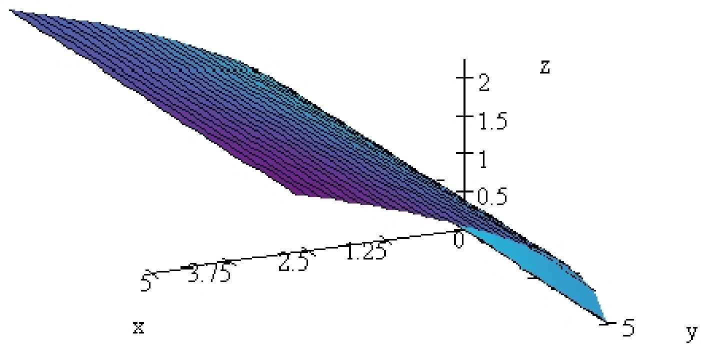
$$\begin{aligned} \vec{r}\,'(t) &= \langle 1, -2\sin t, 2\cos t \rangle \\ \vec{T}(t) &= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 1, -2\sin t, 2\cos t \rangle}{\sqrt{1^2 + (-2\sin t)^2 + (2\cos t)^2}} = \frac{1}{\sqrt{5}} \langle 1, -2\sin t, 2\cos t \rangle \\ \vec{T}\,'(t) &= \frac{1}{\sqrt{5}} \langle 0, -2\cos t, 2\sin t \rangle \\ \vec{N}(t) &= \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|} = \frac{\left\langle 0, -\frac{2}{\sqrt{5}}\cos t, \frac{2}{\sqrt{5}}\sin t \right\rangle}{\sqrt{0^2 + \left(-\frac{2}{\sqrt{5}}\cos t\right)^2 + \left(\frac{2}{\sqrt{5}}\sin t\right)^2}} = \langle 0, -\cos t, \sin t \rangle \\ \vec{N}\left(\frac{\pi}{3}\right) &= \left\langle 0, -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \end{aligned}$$

(b) Find the tangential component of the acceleration vector at $t = \frac{\pi}{3}$.

$$a_T = \frac{d}{dt}\,|\vec{r}\,'(t)| = \frac{d}{dt}\sqrt{5} = 0$$

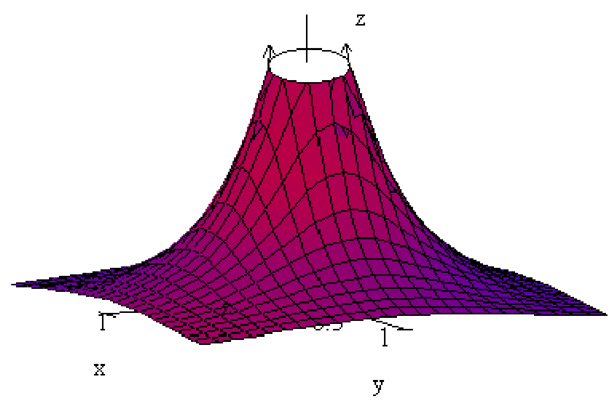
5. Describe and sketch the graph of $z = f(x, y)$ for each of the following functions.

(a) $f(x, y) = \sqrt{x}$



This graph is a has constant cross sections along the $y-$ axis in the shape of a half-parabola.

(b) $f(x, y) = \dfrac{1}{x^2 + y^2}$

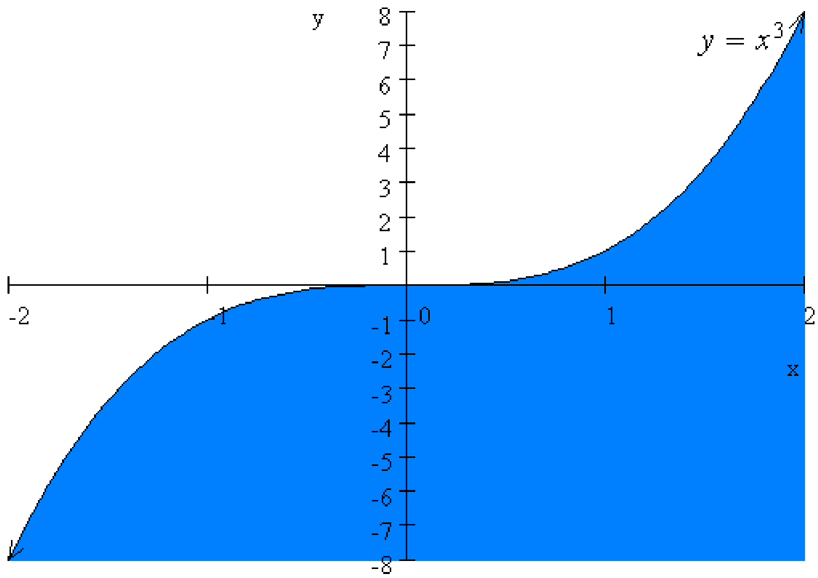


This graph is shaped like a mountain with a single peak that reaches to infinity.

6. Let $f(x,y) = \sqrt{x^3 - y}$.

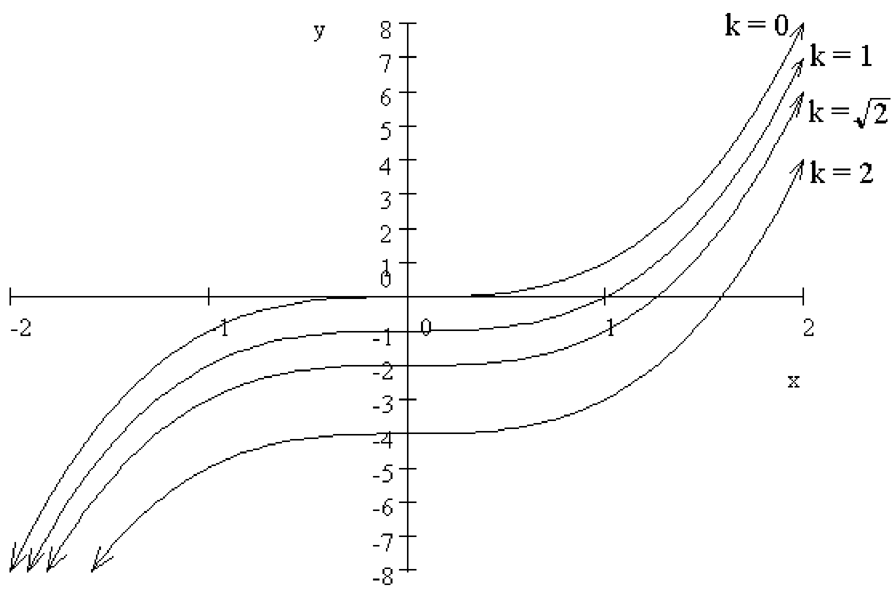
(a) Sketch the domain of $f(x,y)$.

The domain is $x^3 - y \geq 0$, that is, $y \leq x^3$.



(b) Sketch a contour map for $f(x,y)$ showing at least four level curves.

$$\begin{aligned} \sqrt{x^3 - y} &= k \\ x^3 - y &= k^2, \; k \geq 0 \\ y &= x^3 - k, \; k \geq 0 \end{aligned}$$



7. A particular particle moves with acceleration after t seconds given by $\vec{a}(t) = \langle 6t, 4, 5-2t \rangle$ meters/seconds^2. The velocity of the particle at $t = 1$ second is $\langle -9, -4, -2 \rangle$.

(a) Find the initial velocity.

$$\begin{aligned}
\vec{v}(t) &= \int \vec{a}(t)\,dt \\
&= \left\langle 3t^2, 4t, 5t - t^2 \right\rangle + \vec{C} \\
\vec{v}(1) &= \langle 3, 4, 4 \rangle + \vec{C} \text{ and } \vec{v}(1) = \langle -9, -4, -2 \rangle \\
\text{Thus, } \vec{C} &= \langle -12, -8, -6 \rangle. \\
\vec{v}(t) &= \left\langle 3t^2 - 12, 4t - 8, 5t - t^2 - 6 \right\rangle \\
\vec{v}(0) &= \langle -12, -8, -6 \rangle \text{ m/s}
\end{aligned}$$

(b) Determine if the particle is ever at rest, and if so, at what time(s)?

$$\begin{aligned}
& \text{The particle is at rest when } \vec{v}(t) = \vec{0}. \\
& \text{Thus, each component must be zero.} \\
4t - 8 &= 0 \text{ only for } t = 2. \\
\vec{v}(2) &= \langle 0, 0, 0 \rangle. \\
\text{The particle is at rest at } t &= 2 \text{ seconds.}
\end{aligned}$$

8. Short Answer

(a) Show that a line has 0 curvature. (Hint: Start with the most general form of a vector function that describes a line.)

$$\begin{aligned}
\text{The general form of a line is } \vec{r}(t) &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle. \\
\vec{r}\,'(t) &= \langle a, b, c \rangle \\
\vec{r}\,''(t) &= \vec{0}. \\
\text{Thus, } \vec{r}\,'(t) \times \vec{r}\,''(t) &= \vec{0}. \\
\text{Since } \kappa(t) &= \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}, \text{ the curvature is 0.}
\end{aligned}$$

(b) Is it possible for two level curves on a contour map of a function to intersect each other? Fully explain your answer.

No!

If two distinct level curves, $z = k_1$ and $z = k_2$, passed through the same point (x_0, y_0) then we would have $f(x_0, y_0) = k_1$ and $f(x_0, y_0) = k_2$. Since f is a function (each set of inputs has a single output), this is impossible.