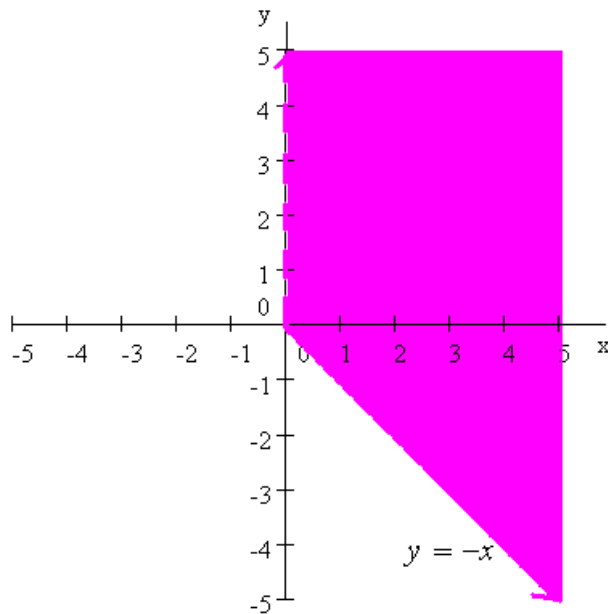


1. Let $f(x, y) = \frac{\sqrt{x+y}}{\sqrt{x}}$.

(a) Sketch the domain of f .

$$\begin{aligned} x + y &\geq 0 \text{ and } x > 0 \\ y &\geq -x \text{ and } x > 0 \end{aligned}$$



(b) Find $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ or show that it does not exist.

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{x+y}}{\sqrt{x}} &= \lim_{r \rightarrow 0} \frac{\sqrt{r \cos \theta + r \sin \theta}}{\sqrt{r \cos \theta}} \\ &= \lim_{r \rightarrow 0} \frac{\sqrt{\cos \theta + \sin \theta}}{\sqrt{\cos \theta}} \\ &= \frac{\sqrt{\cos \theta + \sin \theta}}{\sqrt{\cos \theta}} \end{aligned}$$

Thus, the limit will be different depending on the angle of approach. So $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

(Note: The problem really should have been

$\lim_{(x,y) \rightarrow (0^+, 0^+)} f(x, y)$, since the function is not defined for $x \leq 0$.)

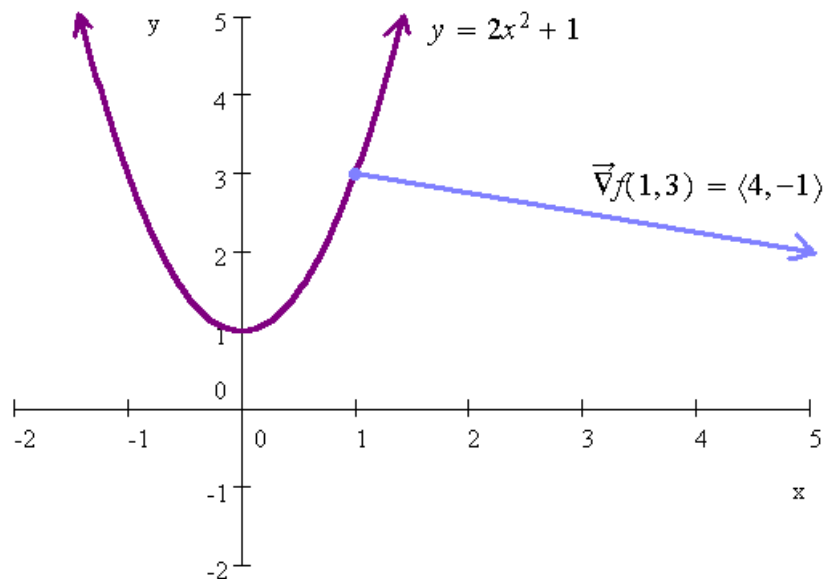
2. Let $f(x, y) = 2x^2 - y + 4$.

(a) Sketch the level curve through the point $(1, 3)$.

$$f(1, 3) = 2(1)^2 - 3 + 4 = 3$$

So, the level curve we want is :

$$\begin{aligned} f(x, y) &= 3 \\ 2x^2 - y + 4 &= 3 \\ y &= 2x^2 + 1 \end{aligned}$$



(b) Find $\vec{\nabla} f(1, 3)$ and add this vector to your sketch above with its tail at $(1, 3)$.

$$\begin{aligned} \vec{\nabla} f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \langle 4x, -1 \rangle \\ \vec{\nabla} f(1, 3) &= \langle 4, -1 \rangle \end{aligned}$$

3. Use differentials and the function $f(x, y) = \frac{x}{\sqrt[3]{y}}$ to estimate $\frac{1.9}{\sqrt[3]{8.12}}$.

$$\Delta x = 1.9 - 2 = -0.1$$

$$\Delta y = 8.12 - 8 = 0.12$$

$$f(x, y) = xy^{-1/3}$$

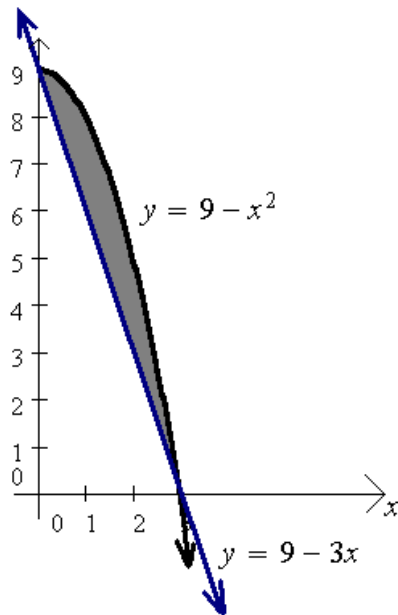
$$f_x(x, y) = y^{-1/3} = \frac{1}{\sqrt[3]{y}}$$

$$f_y(x, y) = -\frac{1}{3}xy^{-4/3} = -\frac{x}{3y\sqrt[3]{y}}$$

$$\begin{aligned}
 dz &= f_x(2, 8) dx + f_y(2, 8) dy \\
 \Delta z &\approx \frac{1}{2}(-0.1) - \frac{1}{24}(0.12) \\
 &= -0.05 - 0.005 \\
 &= -0.055
 \end{aligned}$$

$$\frac{1.9}{\sqrt[3]{8.12}} = f(1.9, 8.12) \approx f(2, 8) + \Delta z \approx 1 - 0.055 = 0.945$$

4. Let $f(x, y) = 3x - xy$. Find the maximum value of f on the following closed region:



$$\begin{aligned}
 f_x(x, y) &= 3 - y \\
 f_y(x, y) &= -x
 \end{aligned}$$

$$\begin{cases} 3 - y = 0 \\ -x = 0 \end{cases} \implies x = 0, y = 3$$

The only critical point is $(0, 3)$, which is not in the region.

Along the straight edge: $y = 9 - 3x, 0 \leq x \leq 3$

$$\begin{aligned}
 f(x, y) &= 3x - x(9 - 3x) \\
 &= 3x^2 - 6x = g(x)
 \end{aligned}$$

$$g'(x) = 6x - 6$$

$$6x - 6 = 0 \implies x = 1 \implies y = 9 - 3(1) = 6 \quad (*)$$

Along the parabolic edge: $y = 9 - x^2, 0 \leq x \leq 3$

$$\begin{aligned}
 f(x, y) &= 3x - x(9 - x^2) \\
 &= x^3 - 6x = h(x)
 \end{aligned}$$

$$h'(x) = 3x^2 - 6$$

$$3x^2 - 6 = 0 \implies x = \pm\sqrt{2} \implies x = \sqrt{2} \implies y = 9 - (\sqrt{2})^2 = 7 \quad (*)$$

In addition to the two points we have so far (marked with an $(*)$), we also have the two intersection points $(0, 9)$ and $(3, 0)$.

(x, y)	$f(x, y)$
$(1, 6)$	-3
$(\sqrt{2}, 7)$	$-4\sqrt{2}$
$(0, 9)$	0
$(3, 0)$	9

So, the maximum value of f over the region is $f(3, 0) = 9$.

5. Let $f(x, y) = 3e^x + x^3y^3 - 3xy$. Show that $(0, 1)$ is a critical point of this function and classify it as a local maximum, local minimum, or saddle point.

$$f_x(x, y) = 3e^x + 3x^2y^3 - 3y$$

$$f_x(0, 1) = 3 - 3 = 0 \quad \checkmark$$

$$f_y(x, y) = 3x^3y^2 - 3x$$

$$f_y(0, 1) = 0 \quad \checkmark$$

So, $(0, 1)$ is a critical point.

$$f_{xy}(x, y) = 9x^2y^2 - 3$$

$$f_{xy}(0, 1) = -3$$

$$f_{xx}(x, y) = 3e^x + 6xy^3$$

$$f_{xx}(0, 1) = 3$$

$$f_{yy}(x, y) = 6x^3y$$

$$f_{yy}(0, 1) = 0$$

$$\begin{aligned} D &= f_{xx}(0, 1)f_{yy}(0, 1) - (f_{xy}(0, 1))^2 \\ &= -9 < 0. \end{aligned}$$

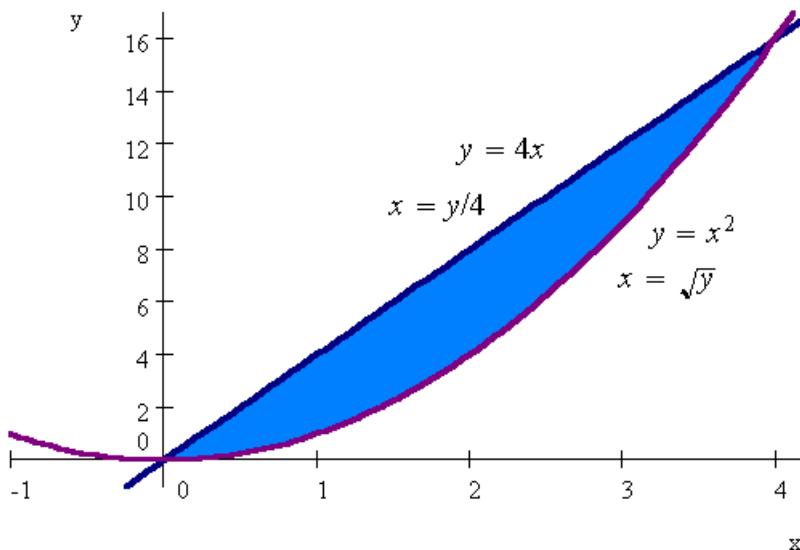
f has a saddle point at $(0, 1)$.

6. (a) Evaluate $\int_0^2 \int_{x+1}^{2x} (e^y + 2xy + 4) dy dx$.

$$\begin{aligned}
 \int_0^2 \int_{x+1}^{2x} (e^y + 2xy + 4) dy dx &= \int_0^2 (e^y + xy^2 + 4y) \Big|_{x+1}^{2x} dx \\
 &= \int_0^2 (e^{2x} + x(2x)^2 + 4(2x)) - (e^{x+1} + x(x+1)^2 + 4(x+1)) dx \\
 &= \int_0^2 e^{2x} - e^{x+1} + 3x^3 - 2x^2 + 3x - 4 dx \\
 &= \frac{1}{2}e^{2x} - e^{x+1} + \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x \Big|_0^2 \\
 &= \left(\frac{1}{2}e^4 - e^3 + \frac{3}{4}(2)^4 - \frac{2}{3}(2)^3 + \frac{3}{2}(2)^2 - 4(2) \right) - \left(\frac{1}{2} - e \right) \\
 &= \frac{1}{2}e^4 - e^3 + e + \frac{25}{6}
 \end{aligned}$$

(b) Reverse the order of integration of the following integral:

$$\int_0^4 \int_{4x}^{x^2} x^3 dy dx$$



$$\int_0^4 \int_{4x}^{x^2} x^3 dy dx = \int_0^{16} \int_{y/4}^{\sqrt{y}} x^3 dx dy$$

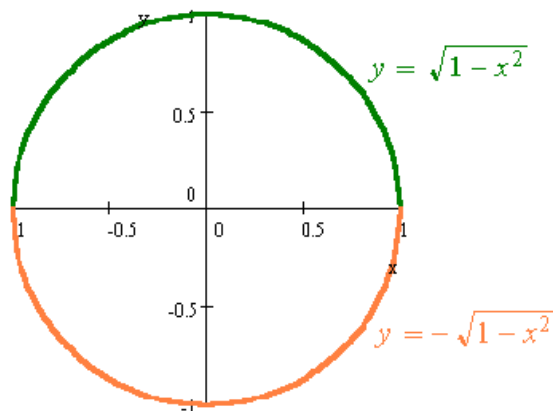
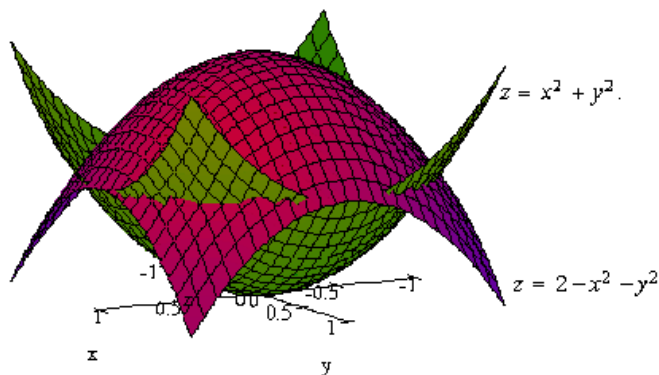
7. Set up, but do not evaluate, a triple integral that represents the volume of the region bounded

above by the paraboloid $z = 2 - x^2 - y^2$ and beneath by the paraboloid $z = x^2 + y^2$.

We need to find the projection of this region on the xy - plane.

In this case, the projection is the intersection of the two surfaces:

$$\begin{aligned} x^2 + y^2 &= 2 - x^2 - y^2 \\ 2x^2 + 2y^2 &= 2 \\ x^2 + y^2 &= 1 \end{aligned}$$



$$V = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=x^2+y^2}^{z=2-x^2-y^2} dz dy dx$$

8. Quick Calculations.

- (a) Suppose that $z = f(x, y)$, $x = g(t)$, and $y = h(t)$. If $g(3) = 4$, $g'(3) = 5$, $h(3) = 6$, $h'(3) = 7$, $f_x(4, 6) = 9$, and $f_y(4, 6) = 11$, find $\frac{dz}{dt}$ when $t = 3$.

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ \frac{dz}{dt} \Big|_{t=3} &= \frac{\partial z}{\partial x} \Big|_{\substack{x=4 \\ y=6}} \frac{dx}{dt} \Big|_{t=3} + \frac{\partial z}{\partial y} \Big|_{\substack{x=4 \\ y=6}} \frac{dy}{dt} \Big|_{t=3} \\ &= f_x(4, 6) g'(3) + f_y(4, 6) h'(3) \\ &= 9 \cdot 5 + 11 \cdot 7 \\ &= 122 \end{aligned}$$

- (b) Let $f(x, y) = 3xy$. Find $D_{\vec{u}}(2, 1)$ where \vec{u} is in the direction of $\langle -3, 4 \rangle$.

$$\begin{aligned} \vec{u} &= \frac{\langle -3, 4 \rangle}{|\langle -3, 4 \rangle|} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \\ \vec{\nabla} f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 3y, 3x \rangle \\ \vec{\nabla} f(2, 1) &= \langle 3, 6 \rangle \end{aligned}$$

$$\begin{aligned}
D_{\vec{u}}(2, 1) &= \vec{\nabla} f(2, 1) \cdot \vec{u} \\
&= \langle 3, 6 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \\
&= 3
\end{aligned}$$

9. Short Answer.

- (a) If the surface $z = f(x, y)$ has a local minimum at the point $(2, 3, 10)$, and the surface has a tangent plane at this point, what is the equation of the tangent plane?

Since the surface has a local minimum at this point,
both partial derivatives are 0.

That is, $f_x(2, 3) = 0$ and $f_y(2, 3) = 0$.

The tangent plane at this point is :

$$\begin{aligned}
f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3) - (z - 10) &= 0 \\
0(x - 2) + 0(y - 3) - (z - 10) &= 0 \\
z &= 10
\end{aligned}$$

(You can also figure this out by picturing a local minimum.

The tangent plane is horizontal at a local minimum, i.e., z is constant.)

- (b) Suppose a mountain climber is trying to quickly get down from a mountain that is shaped like the surface $z = 1000 - x^2y - 4y^2$. If the mountain climber is currently standing at the point $(5, 10, 210)$, which direction (vector) should he head in to descend as quickly as possible?

$$\begin{aligned}
\vec{\nabla} f(x, y) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -2xy, -x^2 - 8y \rangle \\
\vec{\nabla} f(5, 10) &= \langle -100, -105 \rangle
\end{aligned}$$

To descend the quickest, head opposite from the gradient vector. That is, head in the direction of $\langle 100, 105 \rangle$.