1. Let $\overrightarrow{r}(t) = \left\langle \frac{t^2-1}{t+1}, \sqrt{t+10}, t^2 \right\rangle$.

   (a) Find the domain of this function.

   $[-10,-1) \cup (-1,\infty)$
   (That is, all real numbers greater than or equal to -10, except for -1.)

   (b) Find $\lim_{t\to -1} \overrightarrow{r}(t)$.

$$\begin{aligned}\lim_{t\to-1}\left\langle \frac{t^2-1}{t+1}, \sqrt{t+10}, t^2 \right\rangle &= \left\langle \lim_{t\to-1}\frac{t^2-1}{t+1}, \lim_{t\to-1}\sqrt{t+10}, \lim_{t\to-1} t^2 \right\rangle \\ &= \left\langle \lim_{t\to-1}\frac{t^2-1}{t+1}, 3, 1 \right\rangle \\ &\overset{\checkmark}{=} \left\langle \lim_{t\to-1}\frac{2t}{1}, 3, 1 \right\rangle \qquad \text{(the ✓ means I used L'Hospital's Rule)} \\ &=< -2, 3, 1 >\end{aligned}$$

2. Let $\overrightarrow{r}(t) = 3t\overrightarrow{i} + 5t^2\overrightarrow{j} + 7\overrightarrow{k}$.

   (a) Find the equation of the line tangent to this curve when $t = 2$.

   First, we need a point on the line.

$$\overrightarrow{r}(2) = 6\overrightarrow{i} + 20\overrightarrow{j} + 7\overrightarrow{k}$$

   So, (6,20,7) is a point on the tangent line. Now we need a direction vector for the line.

$$\overrightarrow{r}\,'(t) = 3\overrightarrow{i} + 10t\overrightarrow{j} + 0\overrightarrow{k}$$
$$\overrightarrow{r}\,'(2) = 3\overrightarrow{i} + 20\overrightarrow{j} + 0\overrightarrow{k}$$

   So, <3,20,0>is a direction vector for the tangent line. Thus, the parametric equations for the tangent line are

$$x = 6 + 3t,\ y = 20 + 20t,\ z = 7.$$

   (b) Find the velocity, acceleration, and speed of this curve when $t = 2$.

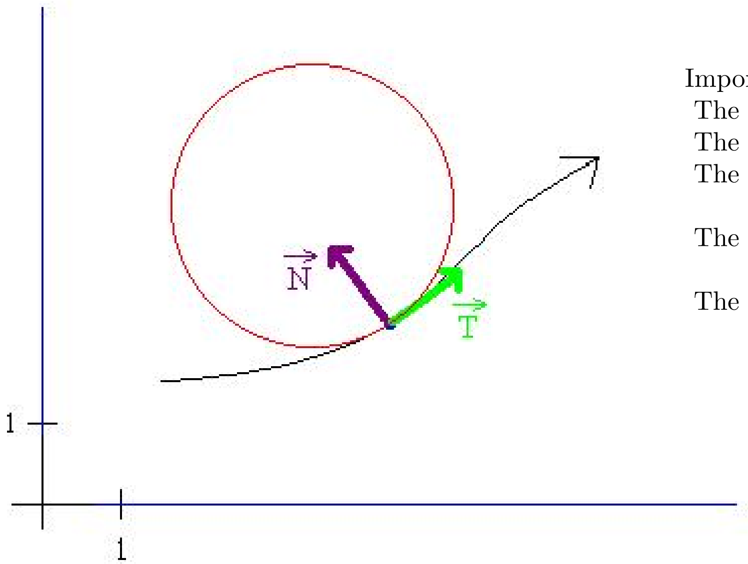
   We found the velocity above. It is $< 3, 20, 0 >$.

   The speed is the magnitude of the velocity, so it is $v(2) = |< 3, 20, 0 >| = \sqrt{409}$

   All we need now is the acceleration.

$$\overrightarrow{a}(t) = \overrightarrow{v}\,'(t) = 0\overrightarrow{i} + 10\overrightarrow{j} + 0\overrightarrow{k}$$
$$\overrightarrow{a}(2) =< 0, 10, 0 >$$

3. For the curve $\overrightarrow{r}(t)$ drawn below, carefully sketch $\overrightarrow{T}$, $\overrightarrow{N}$, and the osculating circle at the point indicated.



Important things to note:
The two vectors are perpendicular.
The normal vector points into the curve.
The tangent vector points in the direction the curve moves.
The osculating circle is tangent to the curve and inside the curve.
The vectors both have length 1.

4. Determine if the curve $\overrightarrow{r}(t) = \langle 2\cos(\pi t), 2\sin(\pi t), t\rangle$ and the cone $x^2 + y^2 = z^2$ intersect (a) never, (b) at all points on the curve, or (c) at a finite set of points (if you determine the correct answer is (c), make sure to give the xyz-coordinates of the points).

Substituting the values of x, y, and z from the curve into the cone, we get :

$$\begin{aligned} (2\cos(\pi t))^2 + (2\sin(\pi t))^2 &= t^2 \\ 4\cos^2(\pi t) + 4\sin^2(\pi t) &= t^2 \\ 4(\cos^2(\pi t) + \sin^2(\pi t)) &= t^2 \\ 4 &= t^2 \\ t &= \pm 2 \end{aligned}$$

So, the curve and cone intersect when $t = 2$, and when $t = -2$.

That is, at the points $(2, 0, 2)$ and $(2, 0, -2)$.

5. Let $f(x, y) = x^2y^3 - e^{xy} + x - 3y^2$.

(a) Find $f_x(2, 1)$ and $f_y(2, 1)$.

$$f_x(x, y) = 2xy^3 - ye^{xy} + 1$$
$$f_x(2, 1) = 4 - e^2 + 1 = 5 - e^2$$

$$f_y(x, y) = 3x^2y^2 - xe^{xy} - 6y$$
$$f_y(2, 1) = 12 - 2e^2 - 6 = 6 - 2e^2$$

(b) Find the equation of the plane tangent to this surface when $x = 2$, and $y = 1$.

First, we need a point on the plane.

$$f(2, 1) = 4 - e^2 + 2 - 3 = 3 - e^2$$

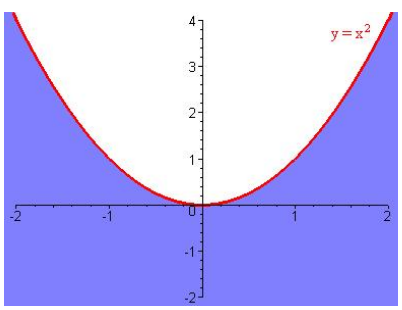
So, $(2, 1, 3-e^2)$ is a point on the tangent plane. Using the derivatives we found in part (a), the equation of the plane is

$$\left(5 - e^2\right)(x - 2) + (6 - 2e^2)(y - 1) = z - (3 - e^2)$$

6. Let $f(x, y) = \sqrt{x^2 - y}$.

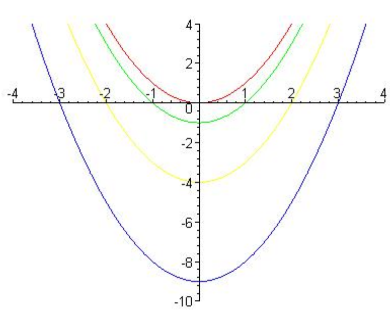
(a) Sketch the domain of this function.

$$x^2 - y \geq 0$$
$$y \leq x^2$$



(b) Draw a contour map for this function, showing.at least 4 level curves.

$$k = 0 \Longrightarrow \sqrt{x^2 - y} = 0 \Longrightarrow y = x^2$$
$$k = 1 \Longrightarrow \sqrt{x^2 - y} = 1 \Longrightarrow y = x^2 - 1$$
$$k = 2 \Longrightarrow \sqrt{x^2 - y} = 2 \Longrightarrow y = x^2 - 4$$
$$k = 3 \Longrightarrow \sqrt{x^2 - y} = 3 \Longrightarrow y = x^2 - 9$$



7. Find $\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4}$, if it exists, or show that the limit does not exist.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4} &= \lim_{r\to 0} \frac{8(r\cos\theta)^2(r\sin\theta)^2}{(r\cos\theta)^4+(r\sin\theta)^4} \\
&= \lim_{r\to 0} \frac{8r^4(\cos\theta)^2(\sin\theta)^2}{r^4((\cos\theta)^4+(\sin\theta)^4)} \\
&= \lim_{r\to 0} \frac{8(\cos\theta)^2(\sin\theta)^2}{((\cos\theta)^4+(\sin\theta)^4)} \\
&= \frac{8(\cos\theta)^2(\sin\theta)^2}{((\cos\theta)^4+(\sin\theta)^4)}
\end{aligned}$$

Since the limit changes depending on the angle of approach, the limit does not exist.

8. Use the curvature formula $\kappa(t) = \frac{|\overrightarrow{r}'(t) \times \vec{r}''(t)|}{|\overrightarrow{r}'(t)|^3}$ to prove that a circle of radius a, lying in the $xy-$plane and centered at the origin has constant curvature.

$$\begin{aligned}
\text{Let } \overrightarrow{r}(t) &= < a\cos t, a\sin t >, 0 \le t \le 2\pi. \\
\text{Then, } \overrightarrow{r}'(t) &= < -a\sin t, a\cos t > \\
\overrightarrow{r}''(t) &= < -a\cos t, -a\sin t > . \\
\text{So, } \overrightarrow{r}'(t) \times \overrightarrow{r}''(t) &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2\overrightarrow{k}.
\end{aligned}$$

$$\text{Thus, } |\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)| = a^2.$$

$$\text{Now, we also have } |\overrightarrow{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a.$$

$$\text{Hence, } \kappa(t) = \frac{|\overrightarrow{r}'(t) \times \vec{r}''(t)|}{|\overrightarrow{r}'(t)|^3} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Therefore, the cuvature is constant (i.e., independent of t).