1. Let $z = xy$, where $x = \frac{s-1}{t+2}$, and $y = \frac{t^2}{s}$. Find $\frac{\partial z}{\partial s}$ at the point where $t = 1$ and $s = 2$.

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= y \cdot \frac{1}{t+2} + x \cdot \frac{-t}{s^2} \\ &= \frac{t^2}{s} \cdot \frac{1}{t+2} + \frac{s-1}{t+2} \cdot \frac{-t}{s^2} \\ \left.\frac{\partial z}{\partial s}\right|_{t=1,s=2} &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{-1}{4} = \frac{1}{12} \end{aligned}$$

2. (a) Verify that $(-1, 0)$ and $(-2, -\frac{1}{2})$ are critical points of $f(x, y) = \frac{1}{4}x^4y^2 - 4xy - 4y$.

$$\begin{aligned} f_x(x, y) &= x^3y^2 - 4y \\ f_y(x, y) &= \frac{1}{2}x^4y - 4x - 4 \end{aligned}$$

$$f_x(-1, 0) = 0 - 0 = 0$$
$$f_y(-1, 0) = 4 - 4 = 0$$

$$f_x(-2, -\tfrac{1}{2}) = -2 + 2 = 0$$
$$f_y(-2, -\tfrac{1}{2}) = -4 + 8 - 4 = 0$$

(b) Classify each of these critical points as either a local maximum, local minimum, or saddle point.

$$\begin{aligned} f_{xx}(x, y) &= 3x^2y^2 \\ f_{xy}(x, y) &= 2x^3y - 4 \\ f_{yy}(x, y) &= \frac{1}{2}x^4 \end{aligned}$$

$$f_{xx}(-1, 0) = 0$$
$$f_{xy}(-1, 0) = -4$$
$$f_{yy}(-1, 0) = \tfrac{1}{2}$$
$$D(-1, 0) = (0)(\tfrac{1}{2}) - (-4)^2 = -16 < 0$$

$(-1, 0)$ is a saddle point.

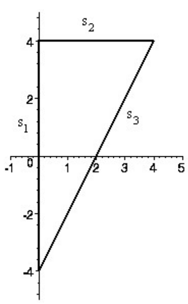
$$f_{xx}(-2, -\tfrac{1}{2}) = 3 > 0$$
$$f_{xy}(-2, -\tfrac{1}{2}) = 4$$
$$f_{yy}(-2, -\tfrac{1}{2}) = 8$$
$$D(-2, -\tfrac{1}{2}) = (3)(8) - (4)^2 = 8 > 0$$

$(-2, -\frac{1}{2})$ is a local minimum.

3. Find the directional derivative of $f(x,y) = x^2 + y^2$ at the point $(2,4)$ heading in the direction of the point $(0,3)$.

$$\begin{aligned}
\overrightarrow{v} &= < 0-2, 3-4 > = < -2, -1 > \\
\overrightarrow{u} &= \frac{\overrightarrow{v}}{|\overrightarrow{v}|} = \frac{< -2, -1 >}{\sqrt{5}} \\
\overrightarrow{\nabla} f(x,y) &= < 2x, 2y > \\
\overrightarrow{\nabla} f(2,4) &= < 4, 8 > \\
D_{\overrightarrow{u}} f(2,4) = \overrightarrow{\nabla} f(2,4) \cdot \overrightarrow{u} &= < 4, 8 > \cdot \frac{< -2, -1 >}{\sqrt{5}} = -\frac{16}{\sqrt{5}}
\end{aligned}$$

4. Find the absolute maximum and minimum of the function $f(x,y) = x^2 - xy + y^2 + 1$ on the closed region R, where R is bounded by the triangle with vertices $(0,4), (0,-4)$, and $(4,4)$.



$$\begin{aligned}
f_x(x,y) &= 2x - y \\
f_y(x,y) &= x + 2y
\end{aligned}
\qquad
\begin{aligned}
2x - y = 0 &\implies \\
x + 2y = 0 &\implies
\end{aligned}
\qquad
\begin{gathered}
y = 2x \\
x + 4x = 0 \\
x = 0 \\
y = 0
\end{gathered}$$

$(0,0)$ is the only critical point.

$$\begin{gathered}
s_1 : x = 0, -4 \le y \le 4 \\
f_1(y) = y^2 + 1 \\
f_1{}'(y) = 2y \\
2y = 0 \\
y = 0 \\
(0,0)
\end{gathered}
\qquad
\begin{gathered}
s_2 : y = 4, 0 \le x \le 4 \\
f_2(x) = x^2 - 4x + 17 \\
f_2{}'(x) = 2x - 4 \\
2x - 4 = 0 \\
x = 2 \\
(2,4)
\end{gathered}
\qquad
\begin{gathered}
s_3 : y = 2x - 4, 0 \le x \le 4 \\
f_3(x) = 3x^2 - 12x + 17 \\
f_3{}'(x) = 6x - 12 \\
6x - 12 = 0 \\
x = 2 \\
(2,0)
\end{gathered}$$

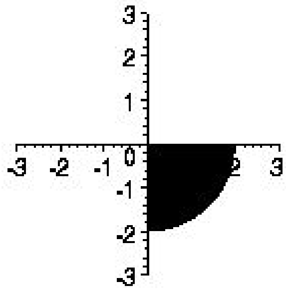
$$\begin{aligned}
f(0,0) &= 1 \\
f(2,4) &= 13 \\
f(2,0) &= 5 \\
f(0,4) &= 17 \\
f(0,-4) &= 17 \\
f(4,4) &= 17
\end{aligned}$$

The absolute minimum is 1.

The absolute maximum is 17.

5. Consider the iterated integral $\int_0^2 \int_{-\sqrt{4-x^2}}^{0} x^2 \, dy \, dx$.

(a) Rewrite, <u>but do not evaluate</u>, this integral with the order of integration switched.

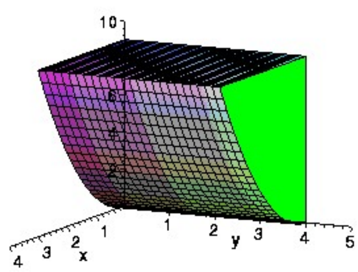


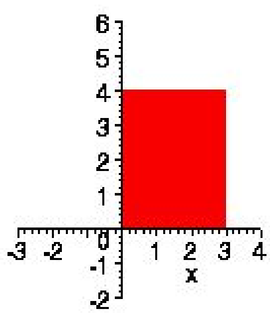
$$\int_{-2}^{0} \int_{0}^{\sqrt{4-y^2}} x^2 \, dx \, dy$$

(b) Rewrite, <u>but do not evaluate</u>, it using polar coordinates.

$$\int_{\frac{3\pi}{2}}^{2\pi} \int_0^2 (r\cos\theta)^2 r \, dr \, d\theta = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^2 r^3 \left(\cos\theta\right)^2 \, dr \, d\theta$$

6. Write, <u>but do not evaluate</u>, a triple integral representing the volume of the region in the 1st octant that is bounded by the parabolic cylinder $z = x^2$, the plane $z = 9$, and the plane $y = 4$.





$$\int_0^3 \int_0^4 \int_{x^2}^{9} dz \, dy \, dx$$

7. Express the point given in rectangular coordinates by $(-\sqrt{3}, 0, 1)$ in:

(a) cylindrical coordinates.

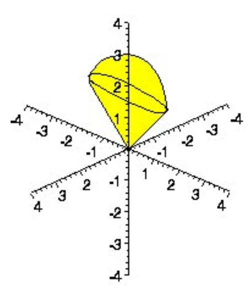
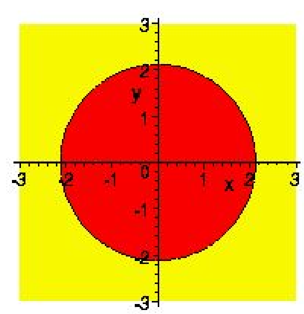
$$r = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2 + 0^2} = \sqrt{3}.$$
$$\tan\theta = \frac{y}{x} = \frac{0}{\sqrt{3}} = 0. \qquad \theta = \pi.$$
$$z = 1.$$
$$(r, \theta, z) = \left(\sqrt{3}, \pi, 1\right)$$

(b) spherical coordinates.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{\left(-\sqrt{3}\right)^2 + 0^2 + 1^2} = 2.$$
$$\theta = \pi.$$
$$\tan\phi = \frac{r}{z} = \frac{\sqrt{3}}{1} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}. \qquad \phi = \frac{\pi}{3}.$$
$$(\rho, \theta, \phi) = \left(2, \pi, \frac{\pi}{3}\right)$$

8. Write, <u>but do not evaluate</u>, a triple integral representing the volume of the region that is above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = 9$ using:

(a) cylindrical coordinates.

$$\int_0^{2\pi}\int_0^{\frac{3}{\sqrt{2}}}\int_r^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

(b) spherical coordinates.

$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{3} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$