Calculus III Review Worksheet

1. Find the equation of the line tangent to the given function at the indicated point.

   (a) $f(x) = x^3 - 2x^2$ at the point where $x = 1$.

   Solution: $f'(x) = 3x^2 - 4x$. $f'(1) = -1$. Also, $f(1) = -1$.
   Therefore, the slope of the line is -1, and a point on it is $(1, -1)$.
   The equation of the line is $y - 1 = -1(x - 1)$, which simplifies to $y = -x + 2$.

   (b) $g(x) = \frac{x}{x+4}$ at the origin.

   Solution: $g'(x) = \frac{4}{(x+4)^2}$. $g'(0) = \frac{1}{4}$.
   Therefore, the slope of the line is $\frac{1}{4}$, and $(0, 0)$ is on it.
   The equation of the line is $y = \frac{1}{4}x$.

   (c) $h(x) = \cos(x^2)$ at the point where $x = \sqrt{\frac{\pi}{6}}$.

   Solution: $h'(x) = -2x\sin(x^2)$. $h'(\sqrt{\frac{\pi}{6}}) = -\sqrt{\frac{\pi}{6}}$. Also, $h(\sqrt{\frac{\pi}{6}}) = \frac{\sqrt{3}}{2}$.
   Therefore, the slope of the line is $-\sqrt{\frac{\pi}{6}}$, and point is $(\sqrt{\frac{\pi}{6}}, \frac{\sqrt{3}}{2})$.
   The equation of the line is $y - \frac{\sqrt{3}}{2} = -\sqrt{\frac{\pi}{6}}(x - \sqrt{\frac{\pi}{6}})$, which simplifies to $y = -\sqrt{\frac{\pi}{6}}x + \sqrt{\frac{\pi}{6}} + \frac{\sqrt{3}}{2}$.

2. Sketch each curve on a set of $xy-$ axes and state its domain and range.

   (a) $f(x) = 6 - x^2$.

   dhF182.9375pt161pt0ptFigure

   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 6]$

   (b) $xy = 1$.

   dhF185.1875pt150.375pt0ptFigure

   Domain: $(-\infty, 0) \cup (0, \infty)$
   Range: $(-\infty, 0) \cup (0, \infty)$

(c) $4x^2 + 9y^2 - 16x + 18y = 11$

This is the equation of a conic section. It will be easier to graph if we complete the squares first.

$$
\begin{aligned}
4x^2 + 9y^2 - 16x + 18y &= 11 \\
4x^2 - 16x + 9y^2 + 18y &= 11 \\
4(x^2 - 4x) + 9(y^2 + 2y) &= 11 \\
4(x^2 - 4x + 4 - 4) + 9(y^2 + 2y + 1 - 1) &= 11 \\
4(x^2 - 4x + 4) - 16 + 9(y^2 + 2y + 1) - 9 &= 11 \\
4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) &= 36 \\
4(x-2)^2 + 9(y+1)^2 &= 36 \\
\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} &= 1
\end{aligned}
$$

So, this is an ellipse centered at $(2, -1)$.
dhF185.1875pt166.25pt0ptFigure
Domain is [-1,5]
Range is [-3,1]

(d) $4x^2 - 9y^2 - 16x - 18y = 29$.

Again, we complete the square.

$$
\begin{aligned}
4x^2 - 9y^2 - 16x + 18y &= 29 \\
4x^2 - 16x - 9y^2 + 18y &= 29 \\
4(x^2 - 4x) - 9(y^2 - 2y) &= 29 \\
4(x^2 - 4x + 4 - 4) - 9(y^2 - 2y + 1 - 1) &= 29 \\
4(x^2 - 4x + 4) - 16 - 9(y^2 - 2y + 1) + 9 &= 29 \\
4(x^2 - 4x + 4) - 9(y^2 - 2y + 1) &= 36 \\
4(x-2)^2 - 9(y-1)^2 &= 36 \\
\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} &= 1
\end{aligned}
$$

So this is a hyperbola centered at $(2, 1)$.
dhF224.5pt179.125pt0ptFigure
Domain is $(-\infty, -1] \cup [5, \infty)$
Range is $(-\infty, \infty)$

(e) The parametric curve described by $x = 2t + 4, y = t - 1$.

Let's eliminate the parameter t.

$$\begin{aligned} x &= 2t + 4 \\ t &= \frac{x-4}{2} \\ y &= \left(\frac{x-4}{2}\right) - 1 \\ y &= \frac{1}{2}x - 3 \end{aligned}$$

So this is a straight line. Because there is no constraint on the parameter t, this is the complete line, not just a segment or ray.
dhF183.6875pt145.8125pt0ptFigure
Domain is $(-\infty, \infty)$
Range is $(-\infty, \infty)$

(f) The parametric curve described by $x = 2 + \cos t, y = 1 + 3\sin t$.

Again, we eliminate the parameter.

$$\begin{aligned} (\cos t)^2 + (\sin t)^2 &= 1 \\ (x-2)^2 + \left(\frac{y-1}{3}\right)^2 &= 1 \\ (x-2)^2 + \frac{(y-1)^2}{9} &= 1 \end{aligned}$$

So this is an ellipse centered at $(2, 1)$.
dhF142.8125pt148.875pt0ptFigure
Domain is $[1, 3]$
Range is $[-2, 4]$

3. Find the area trapped between the curves $y = 3x^2 + 2, x = -1, x = 2$, and $y = 0$.

The area is given by the definite integral $\int_{x=-1}^{x=2} 3x^2 + 2 \ dx$

$$\begin{aligned} \int_{x=-1}^{x=2} 3x^2 + 2 \ dx &= x^3 + 2x]_{x=-1}^{x=2} \\ &= 10 - (-3) \\ &= 13 squareunits \end{aligned}$$

4. Find the shortest distance from the point $(2, 3)$ to the line $y = 2x$. (You should do this using calculus, but if you want to confirm your answer using plain old algebra, go for it.)

Suppose (x, y) is a point on the line. Then the distance from that point to $(2, 3)$ is given by the following formula: $d = \sqrt{(x-2)^2 + (y-3)^2}$. Because the point is on the line, $y = 2x$, the distance can be written as $d = \sqrt{(x-2)^2 + (2x-3)^2}$. Now, we make things easier by realizing that if we find the x-value that minimizes the distance, we will also have the value that minimizes the square of the distance. The square of the distance is given by $d^2 = (x-2)^2 + (2x-3)^2$. This is the function we want to minimize. Recall that we do this by setting its derivative equal to 0.

$$\begin{aligned} f(x) &= (x-2)^2 + (2x-3)^2 \\ f(x) &= x^2 - 4x + 4 + 4x^2 - 12x + 9 \\ &= 5x^2 - 16x + 13 \\ f'(x) &= 10x - 16 \end{aligned}$$

Setting $10x - 16 = 0$, gives $x = 8/5$. Since the function is a parabola that opens upward, this x-value must be a minimum point, not a maximum point. The distance that corresponds to this x-value is $\sqrt{(\frac{8}{5} - 2)^2 + (2(\frac{8}{5}) - 3)^2} = \sqrt{(\frac{-2}{5})^2 + (\frac{1}{5})^2} = \sqrt{\frac{4}{25} + \frac{1}{25}} = \sqrt{\frac{1}{5}}$ units.