1. Calculate each of the following:

(a) $\lim\limits_{x\to\infty} \dfrac{5x^2 - 6x + 4}{2x^2 + 3}$

$$\lim_{x\to\infty} \frac{5x^2-6x+4}{2x^2+3} = \lim_{x\to\infty} \frac{x^2\left(5 - \frac{6}{x} + \frac{4}{x^2}\right)}{x^2\left(2 + \frac{3}{x^2}\right)} = \lim_{x\to\infty} \frac{5 - \overset{\overset{0}{\uparrow}}{\frac{6}{x}} + \overset{\overset{0}{\uparrow}}{\frac{4}{x^2}}}{2 + \underset{\underset{0}{\downarrow}}{\frac{3}{x^2}}} = \frac{5}{2}$$

(b) $\lim\limits_{x\to-\infty} \dfrac{x + x^3}{x + x^2}$

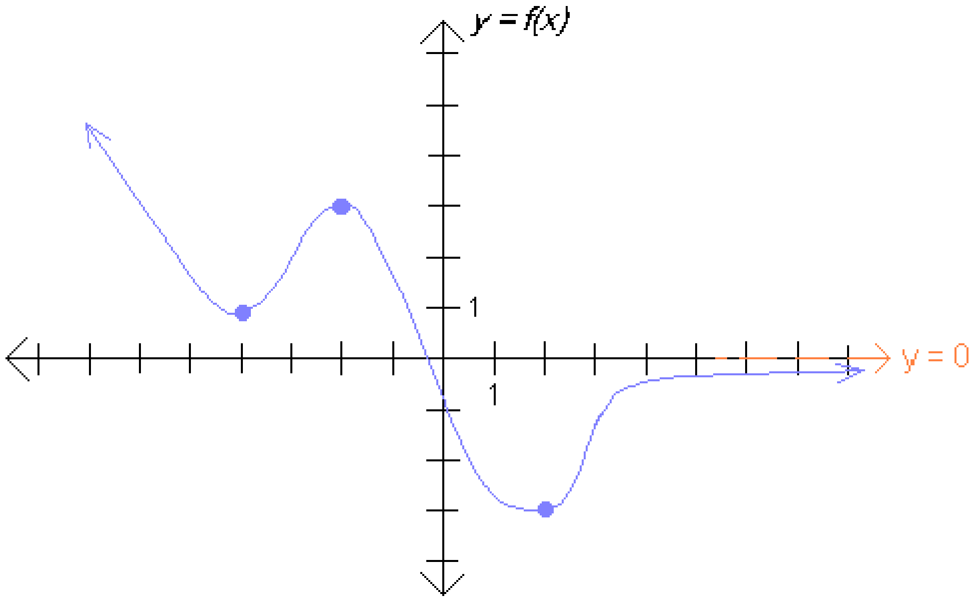
$$\lim_{x\to-\infty} \frac{x+x^3}{x+x^2} = \lim_{x\to-\infty} \frac{x^3\left(\frac{1}{x^2}+1\right)}{x^2\left(\frac{1}{x}+1\right)} = \lim_{x\to-\infty} \frac{\overset{\overset{-\infty}{\uparrow}}{x}\ (\overset{\overset{0}{\uparrow}}{\frac{1}{x^2}}+1)}{\underset{\underset{0}{\downarrow}}{\frac{1}{x}}+1} = -\infty$$

(c) $\int\limits_{x=-2}^{x=2} 9x^2 - 3x + 4\ dx$

$$\begin{aligned}
\int_{x=-2}^{x=2} 9x^2 - 3x + 4\ dx &= 3x^3 - \frac{3}{2}x^2 + 4x\Big]_{x=-2}^{x=2} \\
&= \left(3\,(2)^3 - \frac{3}{2}\,(2)^2 + 4\,(2)\right) - \left(3\,(-2)^3 - \frac{3}{2}\,(-2)^2 + 4\,(-2)\right) \\
&= (24 - 6 + 8) - (-24 - 6 - 8) \\
&= 64
\end{aligned}$$

2. Sketch a curve that satisfies all of the following:

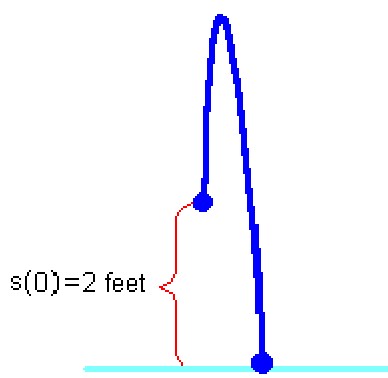
$$\text{dom} f = (-\infty, \infty)\,, f(-2) = 3, f(2) = -3, \lim_{x\to\infty} f(x) = 0$$
$$f'(x) > 0 \text{ for } -4 < x < -2 \text{ and } x > 2$$
$$f'(x) < 0 \text{ for } x < -4 \text{ and } -2 < x < 2$$
$$f''(x) > 0 \text{ for } x < -3 \text{ and } -1 < x < 3$$
$$f''(x) < 0 \text{ for } -3 < x < -1 \text{ and } x > 3$$



3. (a) Let $f'(x) = x^2 - 6x + 2$ and $f(3) = 2$. Find $f(1)$.

$$\begin{aligned} f'(x) &= x^2 - 6x + 2 \\ f(x) &= \frac{1}{3}x^3 - 3x^2 + 2x + C \\ & \quad f(3) = \frac{1}{3}(3)^3 - 3(3)^2 + 2(3) + C = -12 + C \\ & \quad \text{On the other hand, } f(3) = 2. \text{ So } C = 14. \\ f(x) &= \frac{1}{3}x^3 - 3x^2 + 2x + 14 \\ f(1) &= \frac{1}{3}(1)^3 - 3(1)^2 + 2(1) + 14 \\ &= \frac{1}{3} - 3 + 2 + 14 \\ &= \frac{40}{3} \end{aligned}$$

(b) A child holds a stone 2 feet above the surface of a pond. He tosses the stone upward and lets it land in the water. The vertical velocity of the stone is given by $v(t) = 3 - 4t$ feet per second where t is the number of seconds since the stone was thrown. When does the stone hit the water?



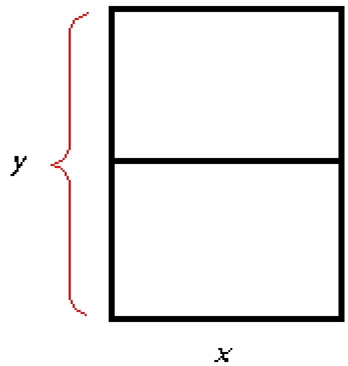
$$\begin{aligned} v(t) &= 3 - 4t, s(0) = 2 \\ s(t) &= 3t - 2t^2 + C \\ & \quad s(0) = 3(0) - 2(0)^2 + C = C \\ & \quad \text{On the other hand, } s(0) = 2. \text{ So } C = 2. \\ s(t) &= 3t - 2t^2 + 2 \end{aligned}$$

Now, we want to know when the stone hits the water. That is, when $s(t) = 0$.

$$\begin{aligned} 3t - 2t^2 + 2 &= 0 \\ -2t^2 + 3t + 2 &= 0 \\ 2t^2 - 3t - 2 &= 0 \\ (2t + 1)(t - 2) &= 0 \\ t = -\frac{1}{2}, t = 2 & \end{aligned}$$

The stone hits the water after 2 seconds.

4. Suppose that you wish to make the ordinary wooden window frame pictured below. If you want to have a total area of 24 square feet, what is the least amount of wood (in feet) you can use? (You must prove that your solution is the least amount of wood, using the techniques developed in class.)



Constraint Equation :

$$\begin{aligned} xy &= 24 \\ y &= \frac{24}{x} \end{aligned}$$

Function to be Optimized :

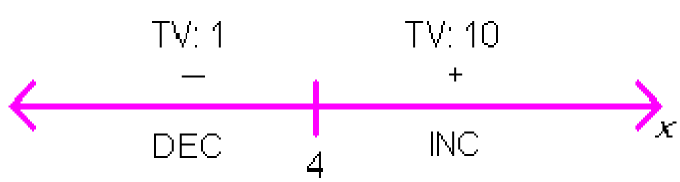
$$\begin{aligned} W &= 3x + 2y \\ W &= 3x + 2\left(\frac{24}{x}\right) \\ W(x) &= 3x + \frac{48}{x} \end{aligned}$$

$$W'(x) = 3 - \frac{48}{x^2}$$

$$\begin{aligned} W'(x) &= 0 \\ 3 - \frac{48}{x^2} &= 0 \\ 3 &= \frac{48}{x^2} \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

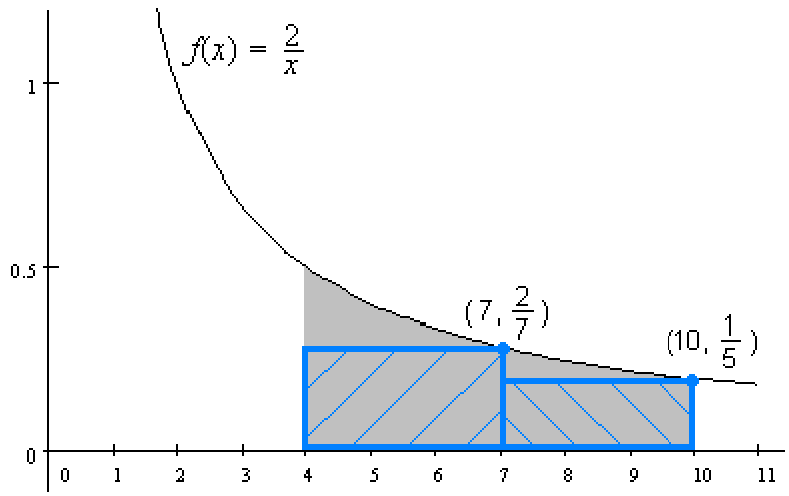
$$\begin{aligned} &W'(x) \text{ DNE} \\ &3 - \frac{48}{x^2} \text{ DNE} \\ &x = 0 \end{aligned}$$

$x = 4$ is the only critical number in the domain $(0 < x < \infty)$.



So, $W(x)$ does have its minimum value when $x = 4$. That minimum value is $W(4) = 3(4) + \frac{48}{4} = 24$ feet.

5. Consider the following shaded region:



(a) Estimate the area the region by finding R_2.

$$R_2 = (3)\left(\frac{2}{7}\right) + (3)\left(\frac{1}{5}\right) = \frac{6}{7} + \frac{3}{5} = \frac{51}{35}$$

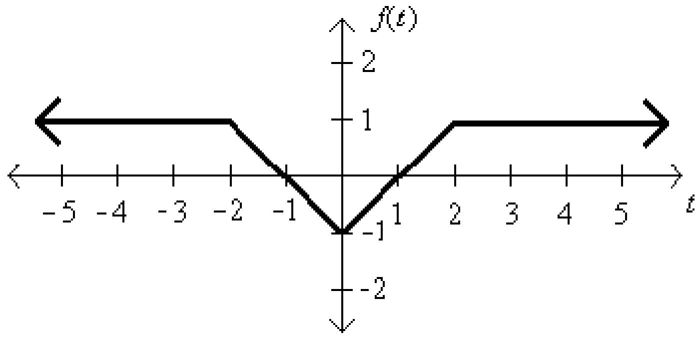
(b) Write a definite integral that represents the exact area of this region.

$$\text{Area} = \int_{x=a}^{x=b} f(x)\,dx = \int_{x=4}^{x=10} \frac{2}{x}\,dx$$

(c) Write a limit (of a Riemann sum) expression that represents the exact area of this region.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{i=n} f(a + i\frac{b-a}{n}) \cdot \frac{b-a}{n} = \lim_{n\to\infty} \sum_{i=1}^{i=n} \frac{2}{4 + i\left(\frac{6}{n}\right)} \cdot \frac{6}{n}$$

6. Let $f(t)$ be the curve drawn below and let $g(x) = \int_{t=-2}^{t=x} f(t)dt$.

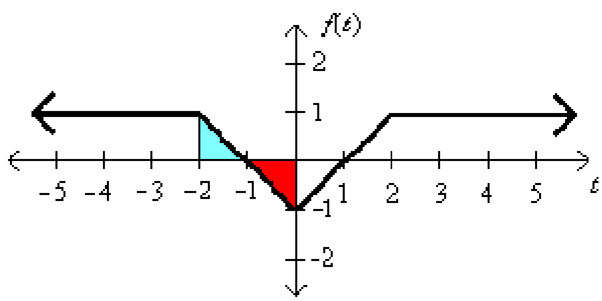


(a) Find $g(-2)$.

$$g(-2) = \int_{t=-2}^{t=-2} f(t)dt = 0$$

(b) Find $g(0)$.

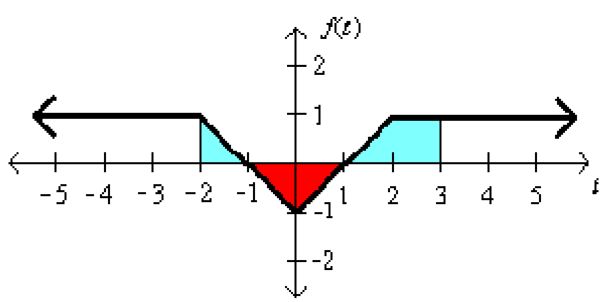
$$g(0) = \int_{t=-2}^{t=0} f(t)dt$$



$$\frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) = 0$$

(c) Find $g(3)$.

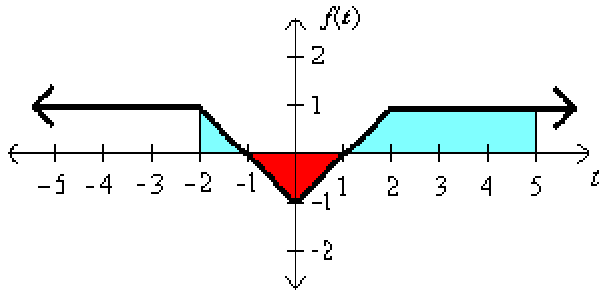
$$g(3) = \int_{t=-2}^{t=3} f(t)dt$$



$$\frac{1}{2}(1)(1) - \frac{1}{2}(2)(1) + \frac{1}{2}(1)(1) + (1)(1) = 1$$

(d) Find $g(5)$.

$$g(5) = \int_{t=-2}^{t=5} f(t)dt$$



$$\frac{1}{2}(1)(1) - \frac{1}{2}(2)(1) + \frac{1}{2}(1)(1) + (3)(1) = 3$$

(e) Find $g'(3)$.

$$\begin{aligned} g'(x) &= f(x) \\ f'(3) &= f(3) = 1. \end{aligned}$$