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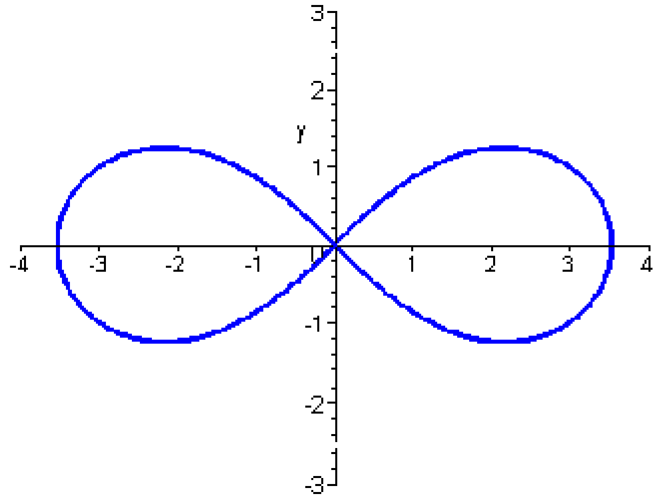
The lemniscate looks like this:



To find $\dfrac{dy}{dx}$:

$$\begin{aligned}
2\left(x^2+y^2\right)^2 &= 25\left(x^2-y^2\right)\\
\frac{d}{dx}\left(2\left(x^2+y^2\right)^2\right) &= \frac{d}{dx}\left(25\left(x^2-y^2\right)\right)\\
4\left(x^2+y^2\right)\left(2x+2y\frac{dy}{dx}\right) &= 25\left(2x-2y\frac{dy}{dx}\right)\\
&\text{For convenience we will stop here.}
\end{aligned}$$

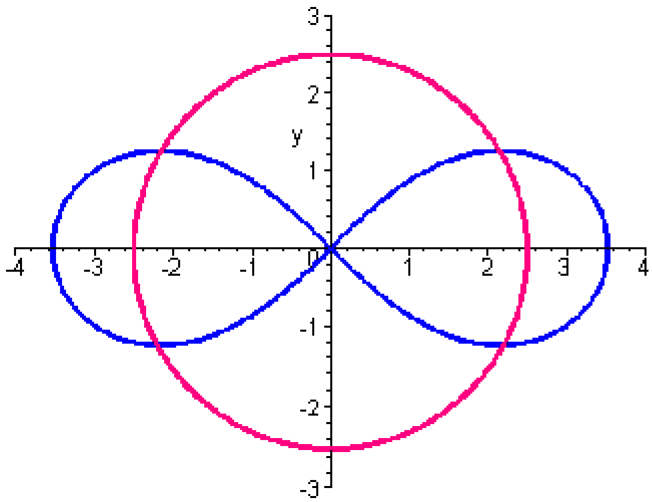
Now, since we want to find the points (x, y) where the tangent line is horizontal, we want to know where $\dfrac{dy}{dx} = 0$. So we will set $\dfrac{dy}{dx} = 0$ in the equation above.

$$\begin{aligned}
4\left(x^2+y^2\right)(2x+2y(0)) &= 25(2x-2y(0))\\
4\left(x^2+y^2\right)(2x) &= 25(2x)\\
8x\left(x^2+y^2\right) &= 50x
\end{aligned}$$

Now, note that if $x = 0$, then $y = 0$ (from the original equation). But at $(0, 0)$, you can see from the picture of the lemniscate that the slope is not 0. Thus, $x = 0$ is not possible for the points we are trying to find. So, we can divide by x.

$$\begin{aligned}
8x\left(x^2+y^2\right) &= 50x\\
8\left(x^2+y^2\right) &= 50\\
x^2+y^2 &= \frac{25}{4}
\end{aligned}$$

This last equation is a circle centered at the origin with radius $\sqrt{\dfrac{25}{4}} = \dfrac{5}{2}$. Thus, the points we are interested in all lie on this circle (as well as on the lemniscate, of course).



You can see from this picture that there are four points that lie on both the lemniscate and the circle. Let's find them. We need to solve simultaneously the two equations

$$2\left(x^2+y^2\right)^2 = 25\left(x^2-y^2\right) \quad \text{AND} \quad x^2+y^2=\frac{25}{4}$$

$$\text{The second equation is the same as } y^2 = \frac{25}{4}-x^2$$

Using substitution :

$$\begin{aligned}
2\left(\frac{25}{4}\right)^2 &= 25\left(x^2-\left(\frac{25}{4}-x^2\right)\right)\\
\frac{625}{8} &= 25\left(2x^2-\frac{25}{4}\right)\\
\frac{625}{8} &= 50x^2-\frac{625}{4}\\
50x^2 &= \frac{1875}{8}\\
x^2 &= \frac{75}{16}\\
x &= \pm\sqrt{\frac{75}{16}}\\
x &= \pm\frac{5\sqrt{3}}{4}
\end{aligned}$$

Now, since $y^2=\frac{25}{4}-x^2$, we have:

$$\begin{aligned}
y &= \pm\sqrt{\frac{25}{4}-x^2}\\
&= \pm\sqrt{\frac{25}{4}-\frac{75}{16}}\\
&= \pm\sqrt{\frac{25}{16}}\\
&= \pm\frac{5}{4}
\end{aligned}$$

Thus, our four points turn out to be : $\left(\frac{5\sqrt{3}}{4},\frac{5}{4}\right),\left(\frac{5\sqrt{3}}{4},-\frac{5}{4}\right),\left(-\frac{5\sqrt{3}}{4},\frac{5}{4}\right)$, and $\left(-\frac{5\sqrt{3}}{4},-\frac{5}{4}\right)$

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First we must find where the curve crosses the $x-$axis. That is, find the $x-$values when $y=0$. Setting $y=0$ in the equation gives us

$$\begin{aligned}
x^2-x(0)+0^2 &= 3\\
x^2 &= 3\\
x &= \pm\sqrt{3}
\end{aligned}$$

Thus, the two points we are interested in are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$. We wish to show that the tangent lines at these two points are parallel (that is, have the same slope). So we will start by finding the slopes.

$$\begin{aligned}
x^2 - xy + y^2 &= 3 \\
\frac{d}{dx}\left(x^2 - xy + y^2\right) &= \frac{d}{dx}(3) \\
2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} &= 0 \\
2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} - x\frac{dy}{dx} &= y - 2x \\
\frac{dy}{dx}(2y - x) &= y - 2x \\
\frac{dy}{dx} &= \frac{y - 2x}{2y - x}
\end{aligned}$$

The slope at the point $(\sqrt{3}, 0)$ is:

$$m_1 = \left.\frac{dy}{dx}\right|_{\substack{x=\sqrt{3}\\ y=0}} = \frac{0 - 2\left(\sqrt{3}\right)}{2(0) - \sqrt{3}} = \frac{-2\sqrt{3}}{-\sqrt{3}} = 2$$

The slope at the point $(-\sqrt{3}, 0)$ is:

$$m_2 = \left.\frac{dy}{dx}\right|_{\substack{x=-\sqrt{3}\\ y=0}} = \frac{0 - 2\left(-\sqrt{3}\right)}{2(0) - \left(-\sqrt{3}\right)} = \frac{2\sqrt{3}}{\sqrt{3}} = 2$$

Since the slopes are the same, the tangent lines are parallel.