1. (a) Find all rational solutions to the equation $x^{1000} - 2x - 3 = 0$.

The Rational Zeroes Theorem gives the only candidates for rational solutions as $\pm 1, \pm 3$
Plugging each of these four values into the equation, we see that $x = -1$ is the only value of these that is a solution.

Thus, $x = -1$ is the only rational solution to the equation.

(b) Find the infimum and supremum of the set $\{x^2 : |x| < 3\}$.

The set is the (open) interval $(0, 9)$.
The infimum is 0. The supremum is 9.

(c) Find the limit of the sequence (s_n) where $s_n = \dfrac{3n}{4n+1}$.

$$\lim_{n\to\infty} \frac{3n}{4n+1} = \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}} = \frac{3}{4}$$

2. For each set of requirements, either give an example that fits the requirements or explain why it is impossible to do so.

(a) Real numbers a, b such that $|a+b| < |a| + |b|$.

$$a = -1, b = 1$$

(b) A bounded, nonempty set $S \subset \mathbb{R}$ with $\sup S > \max S$.

This is impossible. If $\max S$ exists then $\max S = \sup S$ always.

(c) A sequence (s_n) such that s_n converges and $\forall n \in \mathbb{N},\ s_{n+1} > s_n$

$$s_n = 1 - \frac{1}{n}$$

3. Let F be an ordered field. Prove directly from the Field and Order Properties: If $-a \leq -b$ then $b \leq a$. (Make sure to fully justify each step with the appropriate property.)

Proof: Premises: $-a \leq -b$

$-a \leq -b$
$-a + a \leq -b + a$ by O4
$a + (-a) \leq -b + a$ by A2
$0 \leq -b + a$ by A4
$0 \leq a + (-b)$ by A2
$0 + b \leq (a + (-b)) + b$ by O4
$b \leq a + (-b + b)$ by A3 and A1
$b \leq a + (b + (-b))$ by A2
$b \leq a + 0$ by A4
$b \leq a$ by A3

4. Determine if each of the following are TRUE or FALSE. If you determine a statement is false, provide a counterexample. If you determine a statement is true, explain your reasoning.

(a) $\forall n \in \mathbb{N},\ n + 1 \in \mathbb{N}$.

TRUE. This is one of the Peano Axioms.

(b) If S, T are nonempty, bounded sets of real numbers and $S \subset T$, then $\sup(S \cup T) = \sup T$.

TRUE. Since $S \subset T$, we have $S \cup T = T$.

(c) If a sequence (s_n) converges then the sequence $\left(\frac{1}{s_n}\right)$ diverges.

FALSE. Counterexample: $s_n = \frac{n}{n+1}$

5. Suppose $a, b \in \mathbb{R}$ and $a < b$. Circle all of the following that <u>must</u> be true.

$\boxed{\lvert a \cdot (-b) \rvert = \lvert a \rvert \cdot \lvert b \rvert}$	$a^2 < b^2$	$\boxed{\exists q \in \mathbb{Q} \text{ such that } a < q < b}$
$\exists n \in \mathbb{N}$ such that $a < n < b$	$\exists n \in \mathbb{N}$ such that $n \cdot a > b$	$\sup \{s \cdot t : s < a, t < b\} = a \cdot b$
$\boxed{\sup \{s + t : s < a, t < b\} = a + b}$	$\lim\limits_{n\to\infty} \frac{n}{a-b} = \infty$	$\boxed{\lim\limits_{n\to\infty} (b \cdot n - a \cdot n) = \infty}$

6. Let $S \subset \mathbb{R}^+$ be nonempty and bounded. Let $T = \{s^2 : s \in S\}$. Prove: $\sup T = (\sup S)^2$.

Proof: $\boxed{\text{Premises: } S \subset \mathbb{R}^+ \text{ nonempty,bounded. } T = \{s^2 : s \in S\}.}$

Since S is bounded and nonempty, $\sup S$ exists.
First, we show that $(\sup S)^2$ is an upper bound for T.
Let $t \in T$. Then $t = s^2$ for some $s \in S$.
Now, $s \leq \sup S$, by definiton of supremum.
Thus, $t = s^2 \leq (\sup S)^2$.
Since t was arbitrarily chosen from T, $(\sup S)^2$ is an upperbound for T.
Now, we must show that $(\sup S)^2$ is the <u>least</u> upper bound for T.
Suppose, FSOC, there exists $M \in \mathbb{R}$ such that M is an upper bound for T and $M < (\sup S)^2$. We note that $M > 0$ since $T \subset \mathbb{R}^+$.
Then (because both M and $(\sup S)^2$ are positive), $\sqrt{M} < \sup S$.
Thus, $\exists s \in S$ such that $\sqrt{M} < s$.
Then $M < s^2$.
But $s^2 \in T$.
$\Longrightarrow \Longleftarrow$ (M is an upper bound for T, so no element of T can be larger than it.)
$\therefore (\sup S)^2 = \sup T$