1. For each sequence below:

   i. Give the set of subsequential limits.

   ii. For each element in the set in (i), state a formula for n_k such that (s_{n_k}) has that limit.

   (a) $s_n = \frac{n}{n+1}$

$$\begin{array}{ll} (i) & \{1\} \\ (ii) & n_k = k \end{array}$$

   (b) $s_n = \cos\left(\frac{\pi n}{2}\right)$

$$\begin{array}{lll} (i) & \{-1, 0, 1\} & \\ (ii) & \text{For } -1: & n_k = 4k+2 \\ & \text{For } 0: & n_k = 4k+1 \\ & \text{For } 1: & n_k = 4k \end{array}$$

2. Let (s_n) and (t_n) be bounded sequences such that $\forall n, s_n \leq t_n$. Prove that $\limsup s_n \leq \limsup t_n$.

Proof: Premises: $(s_n), (t_n)$ bounded and $\forall n, s_n \leq t_n$.
Let $N \in \mathbb{R}$.
Let $a = \sup\{s_n : n > N\}$.
Let $\varepsilon > 0$.
Then $a - \varepsilon < a = \sup\{s_n : n > N\}$.
So $\exists m \in \{s_n : n > N\}$ such that $a - \varepsilon < m$.
Thus $m = s_{n_0}$ for some $n_0 > N$.
We have $a < s_{n_0} + \varepsilon \leq t_{n_0} + \varepsilon \leq \sup\{t_n : n > N\} + \varepsilon$.
Since $a < \sup\{t_n : n > N\} + \varepsilon$ holds for arbitrarily small ε,
we have $a \leq \sup\{t_n : n > N\}$.
Thus, $\forall N$, $\sup\{s_n : n > N\} \leq \sup\{t_n : n > N\}$.
Taking the limit on each side as $N \to \infty$ and applying Exercise 9.9(c),
$\lim_{N\to\infty} \sup\{s_n : n > N\} \leq \lim_{N\to\infty} \sup\{t_n : n > N\}$.
$\therefore \limsup s_n \leq \limsup t_n$.

3. Short Answer

   (a) Let $s_n = (-2)^n$. State a formula for n_k such that (s_{n_k}) is a monotonic subsequence of (s_n).

$$n_k = 2k$$

   (b) Let $f(x) = \frac{3}{\sqrt{x}}$ and $g(x) = \left(\frac{5}{x}\right)^2$. Find $g \circ f$ and state its domain.

$$(g \circ f)(x) = g(f(x)) = \left(\frac{5}{\left(\frac{3}{\sqrt{x}}\right)}\right)^2 = \left(\frac{5\sqrt{x}}{3}\right)^2 = \frac{25x}{9} \qquad \text{for } x \geq 0.$$

   (c) Let $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 2 - x & \text{for } 1 \leq x < 3 \\ x + 1 & \text{for } x \geq 3 \end{cases}$ For what $x-$value(s) is f discontinuous?

f is discontinuous at 3 (because $\lim_{x\to 3} f(x)$ does not exist).

(Note : f is continuous at 1, because $\lim_{x\to 1} f(x) = 1 = f(1)$.)

4. Quick Proofs:

(a) Prove that $f(x) = 6x - x^2$ is continuous at -3 directly from the definition of continuity.

Proof: Let (x_n) be in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = -3$.

$$\begin{aligned}\text{Then } \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty}\left(6x_n - (x_n)^2\right)\\ &= \lim_{n\to\infty} 6x_n - \lim_{n\to\infty}(x_n)^2\\ &= 6\lim_{n\to\infty} x_n - \left(\lim_{n\to\infty} x_n\right)^2\\ &= 6(-3) - (-3)^2\\ &= f(-3).\end{aligned}$$

$\therefore f$ is continuous at -3.

(b) Prove that $\exists x \in (1,2)$ such that $7\sin\left(\frac{5\pi}{x}\right) - x = 0$.

Proof: f is continuous on $\mathbb{R} - \{0\}$, so f is continuous on $[1,2]$.
$f(1) = 7\sin(5\pi) - 1 = 0 - 1 = -1$.
$f(2) = 7\sin\left(\dfrac{5\pi}{2}\right) - 2 = 7 - 2 = 5$.
So $f(1) < 0 < f(2)$.
By the Intermediate Value Theorem,
$\exists x \in (1,2)$ such that $7\sin\left(\frac{5\pi}{x}\right) - x = 0$.

5. Prove that $f(x) = 6x - x^2$ is continuous at -3 by verifying the $\varepsilon - \delta$ property.

Proof: Let $\varepsilon > 0$. Let $\delta = \min\left\{1, \dfrac{\varepsilon}{13}\right\}$.
Then, $\forall x \in R$ with $|x - (-3)| < \delta$, we have

$$\begin{aligned}|f(x) - f(-3)| &= \left|6x - x^2 + 27\right|\\ &= \left|x^2 - 6x - 27\right|\\ &= |x+3| \cdot |x-9|\\ &< \delta \cdot |x-9|\\ &< 13\delta \quad (\text{since } -4 < x < -2)\\ &\leq 13 \cdot \frac{\varepsilon}{13}\\ &= \varepsilon.\end{aligned}$$

$\therefore f$ is continuous at -3.

6. For each set of requirements, either give an example that fits the requirements or explain why it is impossible to do so.

(a) A sequence (s_n) with no monotone subsequence.

Impossible. Theorem 11.3 states that every sequence has a monotone subsequence.

(b) A pair of sequences $(s_n), (t_n)$ such that $\limsup s_n t_n < \limsup s_n \cdot \limsup t_n$

$$\begin{aligned} s_n &= (-1)^n\\ t_n &= (-1)^{n+1}\\ \text{Then } \limsup s_n t_n &= -1, \text{ but } \limsup s_n \cdot \limsup t_n = 1 \cdot 1 = 1.\end{aligned}$$

(c) A function f such that $f(2) = -1, f(4) = 1$, but $f(x) \neq 0$ for any $x \in (2,4)$.

$$f(x) = \begin{cases} -1 \text{ for } x < 3\\ 1 \text{ for } x \geq 3\end{cases}$$

7. Let $f(x) = \begin{cases} 2x & \text{if } x \in Q \\ x^2 & \text{if } x \notin Q \end{cases}$.

(a) Prove that f is discontinuous at $x = 5$

Proof: $\forall n \in \mathbb{N}$, let $x_n = 5 + \dfrac{\sqrt{2}}{n}$.
Then $\lim_{n\to\infty} x_n = 5$ every term of (x_n) is irrational.
So
$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (x_n)^2 \\ &= \left(\lim_{n\to\infty} x_n\right)^2 \\ &= 5^2 \\ &= 25 \\ &\neq f(5) = 10. \end{aligned}$$
$\therefore f$ is discontinuous at $x = 5$.

(b) Prove that f is discontinuous at $\sqrt{2}$.

Proof: $\forall n \in \mathbb{N}$, let $t_n = \sqrt{2} + \dfrac{1}{n}$.
$\forall n \in \mathbb{N}$, let x_n be a rational number between $\sqrt{2}$ and t_n.
Then, by the Squeeze Theorem (Exercise 8.5(a)),
we have $\lim_{n\to\infty} x_n = \sqrt{2}$.
Thus
$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} 2x_n \\ &= 2 \lim_{n\to\infty} x_n \\ &= 2\sqrt{2} \\ &\neq f(\sqrt{2}) = 2. \end{aligned}$$
$\therefore f$ is discontinuous at $x = \sqrt{2}$.