1. Determine if each of the following are groups. Clearly justify your answers.

(a) The integers under the operation $a * b = a$.

This is not a group.
There is no identity element.
For example: if $e * 5 = 5$, then e must be 5, but then $e * 6 \neq 6$.

(b) The set $\{1, -1\}$ under ordinary multiplication.

This is a group.
1) It is a binary operation (that is, $a * b$ always returns an element in the set).
2) It is associative (ordinary multiplication is associative).
3) There is an identity element; $e = 1$.
4) Each element has an inverse (the inverse of 1 is 1 and of -1 is -1).

2. Let G be a group. Prove: If $(x * y)^2 = x^2 * y^2$ for all $x, y \in G$, then G is abelian.

Proof. Premise: $\forall x, y \in G$, $(x * y)^2 = x^2 * y^2$.

Let $a, b \in G$ be arbitrarily chosen. Let e be the identity element of G.
Then $(a * b)^2 = a^2 * b^2$.
That is

$$\begin{aligned}
(a * b) * (a * b) &= (a * a) * (b * b) \\
a * b * a * b &= a * a * b * b \\
a^{-1} * a * b * a * b &= a^{-1} * a * a * b * b \\
e * b * a * b &= e * a * b * b \\
b * a * b &= a * b * b \\
b * a * b * b^{-1} &= a * b * b * b^{-1} \\
b * a * e &= a * b * e \\
b * a &= a * b.
\end{aligned}$$

Since a and b were arbitrarily chosen, all elements of G commute with each other.
Conclusion: G is abelian. ■

3. Let G be a group and $x \in G$. Prove: $< x^2 > \subseteq < x >$.

Proof. Premise: None.

Let $a \in < x^2 >$ be arbitrarily chosen.

Then $a = \left(x^2\right)^n$ for some $n \in \mathbb{Z}$.

Thus, $a = x^{2n}$.

Since $n \in \mathbb{Z}$, $2n \in \mathbb{Z}$ as well.

Thus, $a \in < x >$.

Conclusion: $< x^2 > \subseteq < x >$. ■

4. True - False. If the statement is false, clearly explain how you determined this.

(a) The following table represents a group:

$*$	s	t	u	v	w
s	t	u	w	s	v
t	u	v	s	t	w
u	w	s	v	u	t
v	s	t	u	v	w
w	v	w	t	w	s

FALSE.

According to the table, $t * w = w$ and $v * w = w$.

If this were a group, then we could use w^{-1} to show that $t = v$, which is not true.

(b) The set $\{0, 3, 6\}$ is a subgroup of $(\mathbb{Z}_9, +)$.

TRUE, since it is both a subset of Z_9 and a group in its own right.

(c) If G is a group and $x, y \in G$ then $(x * y)^{-1} = x^{-1} * y^{-1}$

FALSE.

$(x * y)^{-1} = y^{-1} * x^{-1}$.

The earlier statement is only true if G is abelian.

(d) $\mathbb{Z}$ under addition is a cyclic group.

TRUE, $(\mathbb{Z}, +) = < 1 >$.

5. Quick Calculations.

(a) Find the order of 3 in the group $(\mathbb{Z}_{12}, \oplus)$

$o(3) = 4$. (Since $3 \oplus 3 \oplus 3 \oplus 3 = 0$ and no smaller positive number of 3's will get you to 0.)

(b) Find the order of 5 in the group $(\mathbb{Z}_8, \oplus)$

$$o(5) = 8.$$

(c) In the group $(\mathbb{Z}, +)$ write the cyclic subgroup generated by 5 using set notation (i.e., using $\{\quad\}$).

You can write either $\{0, \pm 5, \pm 10, \pm 15, \ldots\}$ or $\{5n \mid n \in \mathbb{Z}\}$

6. (a) Give an example of a set S with a binary operation $*$ on it such that $*$ is not commutative.

One possible example is $S = \mathbb{Z},\ a * b = a - b$.
Another is $S = \mathbb{Q}^+, a * b = a \div b$

(b) What is the identity element in the group $(S, \triangle)$ where S is the set of all subsets of the set $\{2, 3, 4, 5\}$.

The identity element is the empty set.
For example, $\{2, 3\} \triangle \{\} = \{\} \triangle \{2, 3\} = \{2, 3\}$.

(c) Find a subgroup of the group $(\mathbb{R}, +)$

One possible example is $(\mathbb{Z}, +)$.
Another is $(\mathbb{Q}, +)$.