1. Prove: If H and K are subgroups of a group G, then $H \cap K$ is a subgroup of G.

Proof. Premises: H and K are subgroups of a group G

First we must show that $H \cap K$ is a nonempty subset of G.

$H \cap K$ is nonempty because $e_G \in H$ and $e_G \in K$, so $e_G \in H \cap K$.

Let $x \in H \cap K$ be PBAC.

Then $x \in H$ (and $x \in K$). Since $H \subseteq G$, we have $x \in G$.

Thus $H \cap K \subseteq G$.

Now, we may make use of the Subgroup Theorem as follows.

Let $a, b \in H \cap K$ be PBAC.

Then $a, b \in H$ and $a, b \in K$.

Since $(H, *)$ and $(K, *)$ are groups , $a * b \in H$ and $a * b \in K$.

Thus, $a * b \in H \cap K$.

Also, since $(H, *)$ and $(K, *)$ are groups , $a^{-1} \in H$ and $a^{-1} \in K$.

Thus $a^{-1} \in H \cap K$.

By the Subgroup Theorem, we have the following conclusion.

Conclusion: $H \cap K$ is a subgroup of G. ■

2. Let G be a group and H be a subgroup of G. Prove: If $a \in H$, then $aH = H$.

Proof. Premises: H is a subgroup of a group G and $a \in H$.

We must show that $aH \subseteq H$ and $H \subseteq aH$.

$(aH \subseteq H)$:

Let $x \in aH$ be PBAC.

Then $x = a * h$ for some $h \in H$.

Since $a \in H$ and $h \in H$ and $(H, *)$ is a group, we have $x \in H$.

Thus, $aH \subseteq H$

$(H \subseteq aH)$:

Let $y \in H$ be PBAC.

We can use group arithmetic to write:

$$\begin{aligned} y &= e * y \\ &= \left(a * a^{-1}\right) * y \\ &= a * \left(a^{-1} * y\right) \end{aligned}$$

Now, $a^{-1} \in H$ (since $a \in H$) and $y \in H$ and $(H, *)$ is a group, so $a^{-1} * y \in H$.

Thus, $y \in aH$.

Therefore $H \subseteq aH$.

Conclusion: $aH = H$. ■

3. Calculations.

(a) Find the order of the element $(2,4,8)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

$$(2,4,8) * (2,4,8) * (2,4,8) * (2,4,8) * (2,4,8) * (2,4,8)$$
$$= (2 \oplus 2 \oplus 2 \oplus 2 \oplus 2 \oplus 2, 4 \oplus 4 \oplus 4 \oplus 4 \oplus 4 \oplus 4, 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8 \oplus 8)$$
$$= (0,0,0) = e_{\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}}$$

Therefore the order of $(2,4,8)$ is 6.

(Note : you can also get this from the theorem that says the order of the element is the LCM of each of the components $2, 4, 8$ in their respective groups.

(b) Let $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 2 & 5 \end{pmatrix}$. Find $P \circ Q$.

$$P \circ Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 5 & 2 \end{pmatrix}$$

4. Calculations.

(a) Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\{(0,1)\}, \{(0,0), (0,1)\}, \{(0,0), (1,0)\}, \mathbb{Z}_2 \times \mathbb{Z}_2$$

(b) Find all right cosets of $< 2 >$ in $\mathbb{Z}_8$.

$$< 2 >= \{0,2,4,6\}$$
$$< 2 > \oplus 1 = \{1,3,5,7\}$$

The cosets are $\{0,2,4,6\}$ and $\{1,3,5,7\}$.

5. Quick Answer.

(a) In S_6, how many right cosets of A_6 are there?

$$[S_6 : A_6] = \frac{|S_6|}{|A_6|} = 2.$$

There are 2 right cosets.

(b) How many subgroups does a group of order 11 have?

Let G be a group of order (size) 11.
We know that $\{e_G\}$ and G are subgroups of G with orders 1 and 11, respectively.
Since 11 is prime, it has no divisors other than 1 and 11, and thus can have no subgroups other than those with size 1 and 11.
The only possible subgroup with order 1 is $\{e_G\}$, and the only possible one with order 11 is G itself.
Thus, G has only two subgroups.

(c) If $f : \mathbb{S}_4 \to T$ is a bijection, what is $|T|$?

$$|T| = |S_4| = 4! = 24.$$

6. Determine if each statement is TRUE or FALSE. If it is true, write TRUE and explain how you determined this. If it is false, write FALSE and give a counterexample.

   (a) Let G be a finite group. If G has an element of order $|G|$ then G is cyclic.

   TRUE

   Let $x \in G$ have order $|G|$.

   $$\text{Then } < x >= \left\{ e_G, x, x*x, x*x*x, ..., \underbrace{x*x*...*x}_{|G|-1 \text{ factors}} \right\}.$$

   The elements of $< x >$ are all distict elements of G and there are $|G|$ of them.

   Thus, $G =< x >$ and is therefore cyclic.

   (b) If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

   FALSE

   Counterexample : Let $G = Z_6, H = \{0, 2, 4\}, K = \{0, 3\}$

   Then H and K are subgroups of G, but $H \cup K = \{0, 2, 3, 4\}$ is not.

7. Determine if each statement is TRUE or FALSE. If it is true, write TRUE and explain how you determined this. If it is false, write FALSE and give a counterexample.

   (a) If $f : S \to T$ and f is one-to-one, then $|S| = |T|$.

   FALSE

   Counterexample : Let $S = Z_2, T = Z_4$, and $f : S \to T$ be given by $f(x) = 2x$.

   Then f is one-to-one, but $|S| \neq |T|$.

   (b) Let G be a group. Let $f : G \to G$ be given by $f(x) = x^{-1}$. Then f is a bijection.

   TRUE

   $f(x)$ is onto because each element has an inverse.

   $f(x)$ is one-to-one because inverses are unique.