1. Let $H \lhd G$ and $K \lhd G$. Let $S = \{h * k \,|\, h \in H, k \in K\}$. Prove: $S \lhd G$. (You do not have to prove that S is a subgroup og G.)

Proof. Premises: $H \lhd G$ and $K \lhd G$. $S = \{h * k \,|\, h \in H, k \in K\}$.

Let $x \in S$ and $g \in G$ be PBAC.

Then $x = h * k$ for some $h \in H$ and $k \in K$.

Thus,

$$\begin{aligned} g * x * g^{-1} &= g * h * k * g^{-1} \\ &= g * h * (g^{-1} * g) * k * g^{-1} \\ &= \left(g * h * g^{-1}\right) * \left(g * k * g^{-1}\right). \end{aligned}$$

Since $H \lhd G$ and $K \lhd G$, we have $g * h * g^{-1} \in H$ and $g * k * g^{-1} \in K$.

Thus, $g * x * g^{-1} \in S$.

Conclusion: $S \lhd G$ ■

2. Let G be given by the following group table:

$*$	e	r	s	t	u	v	w	x
e	e	r	s	t	u	v	w	x
r	r	e	t	s	v	u	x	w
s	s	t	r	e	w	x	v	u
t	t	s	e	r	x	w	u	v
u	u	v	x	w	r	e	s	t
v	v	u	w	x	e	r	t	s
w	w	x	u	v	t	s	r	e
x	x	w	v	u	s	t	e	r

(a) Find $Z(u)$ (the centralizer of u).

$Z(u)$ is the set of elements that commute with u.
From the table we see that $Z(u) = \{e, r, v, u\}$
(For example, r is in the set because $u * r = v$ and $r * u = v$).

(b) Find $\overline{u}$ (the conjugacy class of u).

This is the set of all elements of the form $g * u * g^{-1}$, where $g \in G$.
When we compute all 8 possible products of this form ($e * u * e^{-1}, r * u * r^{-1}$, etc.),
we see that each product simplifies to either u or v. Thus, $\overline{u} = \{u, v\}$.

3. Determine which of the following are homomorphisms. For each homomorphism, determine if it is a monomorphism, an epimorphism, and/or an isomorphism.

(a) $\phi : G \to G \times G$ given by $\phi(x) = (x, x)$.

Is it a homomorphism?

$$\phi(a * b) = (a * b, a * b) = (a, a) * (b, b) = \phi(a) * \phi(b).$$

Yes, it is a homomorphism.

Is it a monomorphism?

$$\begin{aligned} \phi(a) &= \phi(b) \\ (a, a) &= (b, b) \\ a &= b \end{aligned}$$

Yes, it is a monomorphism.

Is it an epimorphism?

Let $(a, b) \in G \times G$, such that $a \neq b$.
Then there is no element $g \in G$ such that $\phi(g) = (a, b)$.
No, it is not an epimorphism.

(b) $\phi : G \times \mathbb{Z}_2 \to G$ given by $\phi((x, 0)) = x$ and $\phi((x, 1)) = x^{-1}$.

Is it a homomorphism?
$\phi((a, 0) * (b, 1)) = \phi(a * b, 1) = (a * b)^{-1} = b^{-1} * a^{-1}$
But, $\phi(a, 0) * \phi(b, 1) = a * b^{-1}$, which is not the same.
No, it is not a homomorphism.

4. Let $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $\phi((a, b)) = a + b$.

(a) Prove that ϕ is an epimorphism. (Note: you must prove that ϕ is a homomorphism first.)

Proof. Premises: $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is given by $\phi((a, b)) = a + b$
First we must prove that ϕ is a homomorphism.
Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ be PBAC.
Then $\phi((a, b) * (c, d)) = \phi((a + c, b + d)) = a + c + b + d$,
and $\phi((a, b)) + \phi((c, d)) = (a + b) + (c + d) = a + c + b + d$.
Thus, ϕ is a homomorphism.
Now, we must show it is onto.
Let $m \in Z$ be PBAC.
Then $(m, 0) \in \mathbb{Z} \times \mathbb{Z}$, and $\phi((m, 0)) = m$.
Thus, ϕ is onto.
Conclusion: ϕ is an epimorphism. ■

(b) Find $\ker(\phi)$.

$$\begin{aligned} \ker(\phi) &= \{(a, b) \in Z \times Z \mid \phi((a, b)) = 0\} \\ &= \{(a, -a) \mid a \in Z\} \end{aligned}$$

5. Quick Calculations.

(a) Find the order of the element $\{2,8\}$ in the factor group $\mathbb{Z}_{12}\,/<6>$

$$\{2,8\} = <6> \oplus 2$$

Since $\underbrace{(<6>\oplus 2)\circledast(<6>\oplus 2)\circledast(<6>\oplus 2)}_{\text{3 factors}} = <6> \oplus (2\oplus 2\oplus 2) = <6> \oplus 6 = <6>$

and $<6>$ is the identity element in $\mathbb{Z}_{12}\,/<6>$, the order is 3.

(b) Suppose $\phi_1 : G \to H$ and $\phi_2 : H \to K$ are homomorphisms. Let $x \in \ker(\phi_1)$, and $\phi = \phi_2 \circ \phi_1$. Find $\phi(x)$.

$$\begin{aligned}\phi(x) &= \phi_2(\phi_1(x)) \\ &= \phi_2(e_H) \text{ since } x \in \ker(\phi_1) \\ &= e_K \text{ (since a homomorphism always sends the first identity element to the second one).}\end{aligned}$$

6. Determine if each statement is TRUE or FALSE. If it is true, write TRUE and explain how you determined this. If it is false, write FALSE and give a counterexample.

(a) $\mathbb{Z}_2\times\mathbb{Z}_2 \cong \mathbb{Z}_4$.

FALSE
These cannot be isomorphic, because Z_4 has an element of order 4 (1), but $\mathbb{Z}_2\times\mathbb{Z}_2$ does not.

(b) $\mathbb{Z} \cong 2\,\mathbb{Z}$.

TRUE
$\phi : Z \to 2Z$ given by $\phi(x) = 2x$ is an isomorphism.

(c) If G is abelian and $H \lhd G$, then $G\,/\,H$ is abelian.

TRUE

$$\begin{aligned}(H*a)\circledast(H*b) &= H*(a*b) \\ &= H*(b*a) \text{ since } G \text{ is abelian} \\ &= (H*b)\circledast(H*a)\end{aligned}$$