Math 361 Name ______KEY______
Quiz 3

1. Let $S = \{n \,|\, n \in \mathbb{Z} \text{ and } n \text{ is even.}\}$. Determine if the given operation describes a binary operation on S. If it is a binary operation, determine if it is also associative. If it is not a binary operation, explain how you determined this.

   (a) $a * b = a - b$

   This is a binary operation, since the difference of two even numbers is even.
   It is not associative, since, for example, $(2-2)-2 = 0-2 = -2$,
   but $2-(2-2) = 2-0 = 2 \neq -2$.

   (b) $a * b = \frac{a}{b}$

   This is not a binary operation. One reason is that the quotient of two integers is not usually an integer.
   Another reason is that $0 \in S$, but $a * 0$ is undefined.

2. Prove: $\forall n \in \mathbb{Z}^+$, $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$, using the Principle of Mathematical Induction.

   Proof. Premises: none.

   $\forall n \in \mathbb{Z}^+$, let $P(n)$ be the statement $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$.
   Then $P(1)$ is the statement $4 = 2(1)(1+1)$ which is clearly true.
   Now, suppose $\exists k \in \mathbb{Z}^+$ such that $P(k)$ is true.
   That is, $4 + 8 + 12 + \ldots + 4k = 2k(k+1)$.
   Adding $4(k+1)$ to both sides of this equation gives us

$$\begin{aligned} 4 + 8 + 12 + \ldots + 4k + 4(k+1) &= 2k(k+1) + 4(k+1) \\ &= 2k^2 + 2k + 4k + 4 \\ &= 2k^2 + 6k + 4 \\ &= 2\left(k^2 + 3k + 2\right) \\ &= 2\,(k+1)\,(k+2) \\ &= 2\,(k+1)\,((k+1)+1) \end{aligned}$$

   Thus, $P(k+1)$ is true.
   By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

   Conclusion: $\forall n \in \mathbb{Z}^+$, $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$. ■

3. Fill in each table below so that it is a group with the stated properties.

(a) b is the identity element and $c^{-1} = a$

$*$	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

(b) Every element is its own inverse.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Here I have used a as the identity element. If you choose a different element as the identity element, your table will look different.