Math 361 Name KEY

Quiz 9

This is a take-home quiz. You may use the class textbook and notes. You are not permitted to work with each other or to get help from the tutors on any of these particular problems.

1. Determine which of the following are homomorphisms. For each homomorphism, determine if it is a monomorphism, an epimorphism, and/or an isomorphism.

   (a) $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_6 \times \mathbb{Z}_8$ given by $\phi((a,b)) = (3a, 2b)$.

   Let's check the homomorphism property:

$$\begin{aligned}\phi\left((u,v) * (w,x)\right) &= \phi\left((u \oplus_2 w, v \oplus_4 x)\right)\\ &= \left(3\left(u \oplus_2 w\right), 2\left(v \oplus_4 x\right)\right)\\ &= \left(3u \oplus_2 3w, 2v \oplus_4 2x\right) \text{ (You can justify this using Discrete Mathematics methods.)}\\ &= (3u, 2v) * (3w, 2x)\\ &= \phi\left((u,v)\right) * \phi\left((w,x)\right)\end{aligned}$$

   So this is a homomorphism. It is not an epimorphism:

$$(2,3) \in \mathbb{Z}_6 \times \mathbb{Z}_8, \text{ but } \nexists (a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_4 \text{ s.t. } \phi\left((a,b)\right) = (2,3)$$

   It is a monomorphism:

   Let $(q,r), (s,t) \in \mathbb{Z}_2 \times \mathbb{Z}_4$ such that $\phi\left((q,r)\right) = \phi\left((s,t)\right)$

   Then $(3q, 2r) = (3s, 2t)$.

   Thus, $3q = 3s$ and $2r = 2t$.

   So $q = s$ and $r = t$, and therefore $(q,r) = (s,t)$.

   (b) Let G be a group, $g \in G$, and $\phi : G \longrightarrow G$ be given by $\phi(x) = g * x$.

   This is not a homomorphism (unless $|G| = 1$, in which case it is a very boring isomorphism.)

   $\phi(a * b) = g * a * b$, but $\phi(a) * \phi(b) = g * a * g * b$, which is not the same.

2. Prove: If $G \cong H$ and $H \cong K$ then $G \cong K$. (Hint: Let ϕ_1 and ϕ_2 be isomorphisms from G to H and from H to K, respectively.)

Proof: Premises: $G \cong H$, $H \cong K$

Let $\phi_1 : G \to H$ and $\phi_2 : H \to K$ be isomorphisms. Let $\phi : G \to K$ be given by $\phi(x) = \phi_2(\phi_1(x))$.

First we will show that ϕ is a homomorphism.

Let $a, b \in G$.

Then

$$\begin{aligned}\phi(a * b) = \phi_2\left(\phi_1(a * b)\right) &= \phi_2\left(\phi_1(a) * \phi_1(b)\right), \text{ since } \phi_1 \text{ is a homomorphism}\\ &= \phi_2\left(\phi_1(a)\right) * \phi_2\left(\phi_1(b)\right), \text{ since } \phi_2 \text{ is a homomorphism}\\ &= \phi(a) * \phi(b).\end{aligned}$$

So ϕ is a homomorphism.

Now we will show that ϕ is a monomorphism (one-to-one).

Let $x, y \in G$ such that $\phi(x) = \phi(y)$.

Then

$$\begin{aligned} \phi_2(\phi_1(x)) &= \phi_2(\phi_1(y)) \\ \phi_1(x) &= \phi_1(y), \text{ since } \phi_2 \text{ is a monomorphism} \\ x &= y, \text{ since } \phi_1 \text{ is a monomorphism} \end{aligned}$$

Thus, ϕ is a monomorphism.

Finally, we will show that ϕ is an epimorphism (onto).

Let $k \in K$. Then $\exists h \in H$ such that $\phi_2(h) = k$, since ϕ_2 is an epimorphism.

Also, $\exists g \in G$ such that $\phi_1(g) = h$, since ϕ_1 is an epimorphism.

Now, $\phi(g) = \phi_2(\phi_1(g)) = \phi_2(h) = k$.

Therefore, ϕ is an epimorphism.

Hence, ϕ is an isomorphism from G to K.

$\boxed{\text{Conclusion: } G \cong K}$