Worksheet on Implicit Differentiation

Find each of the following:

(a) $\frac{d}{dx}x^3$

$3x^2$

(b) $\frac{d}{dx}y^3$

$3y^2\frac{dy}{dx}$

(c) $\frac{d}{dx}\left(x^3+y^3\right)$

$3x^2+3y^2\frac{dy}{dx}$

(d) $\frac{d}{dx}(xy)$

$x\frac{dy}{dx}+y$

(e) $\frac{d}{dx}\left(x^3y^3\right)$

$3x^3y^2\frac{dy}{dx}+3x^2y^3$

(f) $\frac{d}{dx}\frac{x^3}{y^3}$

$$\frac{3y^3x^2-3x^3y^2\frac{dy}{dx}}{y^6}$$

(g) $\frac{d}{dx}\left(x^2+xy+y^4+7\right)$

$2x+x\frac{dy}{dx}+y+4y^3\frac{dy}{dx}$

(h) $\frac{d}{dx}\sqrt{xy}$

$\frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx}+y\right)$

1. Find $\frac{dy}{dx}$ by implicit differentiation.

   (a) $x^3+5xy-6xy^2=10y-y^3+6$

$$\begin{aligned}
\frac{d}{dx}\left(x^3+5xy-6xy^2\right) &= \frac{d}{dx}\left(10y-y^3+6\right)\\
3x^2+5x\frac{dy}{dx}+5y-12xy\frac{dy}{dx}-6y^2 &= 10\frac{dy}{dx}-3y^2\frac{dy}{dx}\\
5x\frac{dy}{dx}-12xy\frac{dy}{dx}-10\frac{dy}{dx}+3y^2\frac{dy}{dx} &= 6y^2-3x^2-5y\\
\left(5x-12xy-10+3y^2\right)\frac{dy}{dx} &= 6y^2-3x^2-5y\\
\frac{dy}{dx} &= \frac{6y^2-3x^2-5y}{5x-12xy-10+3y^2}
\end{aligned}$$

(b) $x + \sqrt{xy} = y$ (This one is a little harder because there are some necessary algebra steps in the middle.)

$$\begin{aligned}
\frac{d}{dx}\left(x + \sqrt{xy}\right) &= \frac{d}{dx} y \\
1 + \frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx} + y\right) &= \frac{dy}{dx} \\
1 + \frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx} + y\right) &= \frac{dy}{dx} \\
1 + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} + \frac{y}{2\sqrt{xy}} &= \frac{dy}{dx} \\
\frac{x}{2\sqrt{xy}}\frac{dy}{dx} - \frac{dy}{dx} &= -1 - \frac{y}{2\sqrt{xy}} \\
\left(\frac{x}{2\sqrt{xy}} - 1\right)\frac{dy}{dx} &= -1 - \frac{y}{2\sqrt{xy}} \\
\frac{dy}{dx} &= \frac{-1 - \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} - 1}
\end{aligned}$$

Note: This can be simplified all the way to $\frac{dy}{dy} = \frac{2\sqrt{xy} + y}{2\sqrt{xy} - x}$ if you want to give it a try.

(c) $xy^5 + x^5y = 1$

$$\begin{aligned}
\frac{d}{dx}\left(xy^5 + x^5y\right) &= \frac{d}{dx}(1) \\
5xy^4\frac{dy}{dx} + y^5 + x^5\frac{dy}{dx} + 5x^4y &= 0 \\
5xy^4\frac{dy}{dx} + x^5\frac{dy}{dx} &= -y^5 - 5x^4y \\
\left(5xy^4 + x^5\right)\frac{dy}{dx} &= -y^5 - 5x^4y \\
\frac{dy}{dx} &= \frac{-y^5 - 5x^4y}{5xy^4 + x^5}
\end{aligned}$$

2. Consider the curve defined by $x^2 + 3xy + y^2 = 4$.

(a) Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}\left(x^2 + 3xy + y^2\right) &= \frac{d}{dx}(4) \\
2x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} &= 0 \\
3x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -2x - 3y \\
(3x + 2y)\frac{dy}{dx} &= -2x - 3y \\
\frac{dy}{dx} &= \frac{-2x - 3y}{3x + 2y}
\end{aligned}$$

(b) Find the values of y when $x = 2$.

$$\begin{aligned}
x^2 + 3xy + y^2 &= 4 \\
2^2 + 3(2)y + y^2 &= 4 \\
4 + 6y + y^2 &= 4 \\
y^2 + 6y &= 0 \\
y(6 + y) &= 0 \\
y &= 0 \text{ OR } y = -6
\end{aligned}$$

(c) Find the slope of each tangent line to the curve when $x = 2$.

At the point $(2, 0)$:

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=0}} = \left.\frac{-2x - 3y}{3x + 2y}\right|_{\substack{x=2\\y=0}} = \frac{-2(2) - 3(0)}{3(2) + 2(0)} = -\frac{2}{3}$$

At the point $(2, -6)$:

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-6}} = \left.\frac{-2x - 3y}{3x + 2y}\right|_{\substack{x=2\\y=-6}} = \frac{-2(2) - 3(-6)}{3(2) + 2(-6)} = -\frac{7}{3}$$

(d) Find the equation of each tangent line when $x = 2$.

At the point $(2, 0)$:

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 0 &= -\frac{2}{3}(x - 2) \\
y &= -\frac{2}{3}x + \frac{4}{3}
\end{aligned}$$

At the point $(2, -6)$:

$$\begin{aligned}
y - (-6) &= -\frac{7}{3}(x - 2) \\
y &= -\frac{7}{3}x - \frac{4}{3}
\end{aligned}$$