Overview of Rates of Change

To find the slope of a secant line to $f(x)$ from the point where $x = a$ to the point where $x = b$, I would:

$$\text{use the formula } \frac{f(b) - f(a)}{b - a}$$

Example: Find the slope of the secant line to $y = 2x^2 + 3x + 1$ from $x = -2$ to $x = 3$.

$$\begin{aligned} \text{slope of secant line} = \frac{f(b) - f(a)}{b - a} &= \frac{f(3) - f(-2)}{3 - (-2)} \\ &= \frac{28 - 3}{5} \\ &= 5 \end{aligned}$$

Another way to say "the slope of the secant line" is:

the average rate of change

To find the slope of a tangent line to $f(x)$ at the point where $x = a$, I would: (make sure you list both methods)

$$\text{use the formula } \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ or use the formula } \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Example: Find the slope of the tangent line to $y = 2x^2 + 3x + 1$ at $x = -2$.

$$\begin{aligned} \text{slope of tangent line} = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} &= \lim_{x \to -2} \frac{(2x^2 + 3x + 1) - 3}{x + 2} \\ &= \lim_{x \to -2} \frac{2x^2 + 3x - 2}{x + 2} \\ &= \lim_{x \to -2} \frac{(x + 2)(2x - 1)}{x + 2} \\ &= \lim_{x \to -2} 2x - 1 \\ &= -5 \end{aligned}$$

OR

$$\begin{aligned} \text{slope of tangent line} = \lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h} &= \lim_{h \to 0} \frac{2(-2 + h)^2 + 3(-2 + h) + 1 - 3}{h} \\ &= \lim_{h \to 0} \frac{8 - 8h + 2h^2 - 6 + 3h - 2}{h} \\ &= \lim_{h \to 0} \frac{2h^2 - 5h}{h} \\ &= \lim_{h \to 0} 2h - 5 \\ &= -5 \end{aligned}$$

Other ways to say "the slope of the tangent line" are:

the instantaneous rate of change OR the derivative

A shorthand way to write the slope of the tangent line to $f(x)$ at the point where $x = a$ is:

$$f'(a)$$

To find the equation of a tangent line to $f(x)$ at the point where $x = a$, I would:

use the point-slope formula for a line with $f'(a)$ as the slope and $(a, f(a))$ as the point

Example: Find the line tangent to $y = 2x^2 + 3x + 1$ at $x = -2$.

$$\begin{aligned} \text{slope} &: \quad f'(-2) = -5 \text{ (from above)} \\ \text{point} &: \quad (-2, 3) \\ \text{line} &: \quad y - 3 = -5(x - (-2)) \Longrightarrow y = -5x - 7 \end{aligned}$$

1. Find the average rate of change of $y = 4x + 1$ from $x = 3$ to $x = 5$.

$$\frac{f(5) - f(3)}{5 - 3} = \frac{21 - 13}{2} = 4$$

2. Find the instantaneous rate of change of $y = 4x + 1$ at $x = 3$.

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{4x + 1 - 13}{x - 3} = \lim_{x \to 3} \frac{4x - 12}{x - 3} = \lim_{x \to 3} \frac{4(x - 3)}{x - 3} = 4$$

3. Find the instantaneous rate of change of $y = 4x + 1$ at $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{4x + 1 - (4a + 1)}{x - a} = \lim_{x \to 3} \frac{4x - 4a}{x - a} = \lim_{x \to a} \frac{4(x - a)}{x - a} = 4$$

4. Find the slope of the tangent line to $y = \frac{1}{x}$ at $x = 2$.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \to 2} \frac{2 - x}{2x(x - 2)} = \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}$$

5. Find the equation of the tangent line to $y = \frac{1}{x}$ at $x = 2$.

$$\text{line:} \quad y - \frac{1}{2} = -\frac{1}{4}(x - 2) \Longrightarrow y = -\frac{1}{4}x + 1$$

6. Find the slope of the tangent line to $y = \frac{1}{x}$ at $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x \to a} \frac{a - x}{ax(x - a)} = \lim_{x \to a} \frac{-1}{ax} = -\frac{1}{a^2}$$

7. Find the equation of the tangent line to $y = \frac{1}{x}$ at $x = a$.

$$\text{line:} \quad y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \Longrightarrow y = -\frac{1}{a^2}x + \frac{2}{a}$$

8. Let $f(x) = x^2 + 2x$.

   (a) Find $f(3)$.

$$f(3) = 3^2 + 2(3) = 15$$

   (b) Find $f'(3)$.

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 5)}{x - 3} = \lim_{x \to 3} x + 5 = 8$$

(c) Find the equation of the tangent line to $f(x) = x^2 + 2x$ at $x = 3$.

$$\text{line:} \quad y - 15 = 8(x - 3) \Longrightarrow y = 8x - 9$$

(d) Find $f(a)$.

$$f(a) = a^2 + 2a$$

(e) Find $f'(a)$.

$$\begin{aligned} f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 + 2(a+h) - (a^2 + 2a)}{h} &= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 + 2a + 2h -}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2 + 2h}{h} \\ &= \lim_{h\to 0} 2a + h + 2 \\ &= 2a + 2 \end{aligned}$$

(f) Find the equation of the tangent line to $f(x) = x^2 + 2x$ at $x = a$.

$$\text{line:} \quad y - \left(a^2 + 2a\right) = (2a + 2)(x - a) \Longrightarrow y = (2a + 2)x - a^2$$