

1. Clearly explain why each of the following statements are true. (Points will be given for both accuracy and clarity.)

(a) $-6 \mid 18$.

$$18 = -6(3) \text{ and } 3 \text{ is an integer.}$$

(b) 24 is composite.

$$24 > 1, 24 \in \mathbb{Z}, 2 \mid 24 \text{ and } 2 \neq 1 \text{ and } 2 \neq 24.$$

(c) $\{x, y\} \in 2^{\{x, y, z\}}$

$$2^{\{x, y, z\}} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

2. Show that each of the following statements are false by demonstrating a counterexample.

(a) If $x \in \mathbb{Z}$ and $4x$ is even, then x is even.

$$4(5) = 20 \text{ which is even, but } 5 \text{ is not even because } 5 = 2(2.5), \text{ and } 2.5 \notin \mathbb{Z}. \text{ So } x = 5 \text{ provides a counterexample.}$$

(b) An integer $x > 0$ if and only if $x^2 > 0$.

A counterexample is provided by letting $x = -1$. Then $x^2 = 1 > 0$, but $x \leq 0$.

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 3$.

Letting $x = 0$, we get a counterexample, because there is no integer that multiplies with 0 to give a product of 3.

3. Prove: $\neg(x \vee (\neg y))$ is logically equivalent to $(\neg x) \wedge y$.

x	y	$\neg x$	$\neg y$	$(\neg x) \wedge y$	$x \vee (\neg y)$	$\neg(x \vee (\neg y))$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	F	T	F

4. Let $x \in \mathbb{Z}$. Prove: If $3x$ is even, then $9x + 1$ is odd.

Suppose $3x$ is even.

Then, $\exists y \in \mathbb{Z}$ such that $3x = 2y$.

So $9x = 6y$,

and thus $9x + 1 = 6y + 1 = 2(3y) + 1$.

Since $3 \in \mathbb{Z}$ and $y \in \mathbb{Z}$, $3y \in \mathbb{Z}$ by the closure property for integer multiplication.

Therefore, $9x + 1$ is odd.

5. Suppose that at a certain college, students are randomly assigned ID codes that consist of 5 digits (0-9) followed by 4 letters (A-Z).

(a) How many different ID codes are possible?

$$10^5 26^4$$

(b) How many have no repeated characters (i.e., digits or letters)?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 26 \cdot 25 \cdot 24 \cdot 23$$

(c) How many have no digits higher than 7?

$$8^5 26^4$$

6. Let $A = \{x \in \mathbb{Z} : 2x > 9\}$ and $B = \{x \in \mathbb{Z} : x > 1\}$. Prove $A \subseteq B$.

Let $a \in A$.

Then $a \in \mathbb{Z}$ and $2a > 9$.

Thus $a > \frac{9}{2}$,

and so $a > 1$.

Therefore $a \in B$.

Since a was chosen arbitrarily, $A \subseteq B$.

7. (a) Write the following sentence using quantifier notation.

There is an integer that is less than the square of every integer.

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x < y^2.$$

(b) Write the negation of the following statement.

$$\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}, x + y = 4.$$

without using the symbols \neg , \nexists , or \forall .

$$\exists x \in \mathbb{N}, \forall y \in \mathbb{Z}, x + y \neq 4.$$