1. Clearly explain why each of the following statements are true. (Points will be given for both accuracy and clarity.)

   (a) $-7 \mid 28$.

   $$28 = -7\,(4) \text{ and 4 is an integer.}$$

   (b) 20 is composite.

   $$20 \in Z, 2|20 \text{ and } 1 < 2 < 20.$$

   (c) $\{d, e\} \in 2^{\{d,e,f\}}$

   $$2^{\{d,e,f\}} = \{\emptyset, \{d\}, \{e\}, \{f\}, \{d, e\}, \{d, f\}, \{e, f\}, \{d, e, f\}\}$$

2. Show that each of the following statements are false by demonstrating a counterexample.

   (a) If $x \in \mathbb{Z}$ and $2x$ is even, then x is even.

   $$\begin{aligned} 2\,(5) &= 10 \text{ which is even, but 5 is not even because } 5 = 2\,(2.5)\,, \text{ and} \\ 2.5 &\notin Z. \text{ So } x = 5 \text{ provides a counterexample.} \end{aligned}$$

   (b) An integer $x > 0$ if and only if $x^2 > 0$.

   A counterexample is provided by letting $x = -1$. Then $x^2 = 1 > 0$, but $x \leq 0$.

   (c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 3$.

   Letting $x = 0$, we get a counterexample, because there is no integer that multiplies with 0 to give a product of 3.

3. Prove: $\neg\,(x \wedge (\neg y))$ is logically equivalent to $(\neg x) \vee y$.

x	y	$\neg x$	$\neg y$	$(\neg x) \vee y$	$x \wedge (\neg y)$	$\neg\,(x \wedge (\neg y))$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	F	T

4. Let $x \in \mathbb{Z}$. Prove: If $3x$ is even, then $9x + 1$ is odd.

   Suppose $3x$ is even.
   Then, $\exists y \in Z$ such that $3x = 2y$.
   So $9x = 6y$,
   and thus $9x + 1 = 6y + 1 = 2(3y) + 1$.
   Since $3 \in Z$ and $y \in Z$, $3y \in Z$ by the closure property for integer multiplication.
   Therefore, $9x + 1$ is odd.

5. Suppose that at a certain college, students are randomly assigned ID codes that consist of 4 digits (0-9) followed by 5 letters (A-Z).

   (a) How many different ID codes are possible?

   $$10^4 26^5$$

   (b) How many have no repeated characters (i.e., digits or letters)?

   $$10 \cdot 9 \cdot 8 \cdot 7 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$$

   (c) How many have no digits higher than 6?

   $$7^4 26^5$$

6. Let $A = \{x \in \mathbb{Z} : 2x > 9\}$ and $B = \{x \in \mathbb{Z} : x > 1\}$. Prove $A \subseteq B$.

Let $a \in A$.
Then $a \in Z$ and $2a > 9$.
Thus $a > \frac{9}{2}$,
and so $a > 1$.
Therefore $a \in B$.
Since a was chosen arbitrarily, $A \subseteq B$.

7. (a) Write the following sentence using quantifier notation.

There is an integer that is less than the square of every integer.

$$\exists x \in Z, \forall y \in Z, x < y^2.$$

(b) Write the negation of the following statement.

$$\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}, x + y = 4.$$

without using the symbols $\neg$, $\nexists$, or $\not\forall$.

$$\exists x \in N, \forall y \in Z, x + y \neq 4.$$