

1. Prove by Mathematical Induction:  $\forall n \in \mathbb{N}, \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$ .

Proof: Clearly  $\frac{1}{1 \cdot 3} = \frac{0+1}{2(0)+3}$ , so the equation holds for  $n = 0$ .

Suppose that the equation holds for some  $k \in \mathbb{N}$ .

Then  $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$ .

$$\begin{aligned} \text{So } \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2(k+1)+1)(2(k+1)+3)} &= \frac{k+1}{2k+3} + \frac{1}{(2(k+1)+1)(2(k+1)+3)} \\ &= \frac{k+1}{2k+3} + \frac{1}{(2k+3)(2k+5)} \\ &= \frac{(k+1)(2k+5)+1}{(2k+3)(2k+5)} \\ &= \frac{2k^2+7k+6}{(2k+3)(2k+5)} \\ &= \frac{(2k+3)(k+2)}{(2k+3)(2k+5)} \\ &= \frac{k+2}{2k+5} \\ &= \frac{(k+1)+1}{2(k+1)+3} \end{aligned}$$

Thus, the equation holds for  $k + 1$ .

$$\therefore \forall n \in \mathbb{N}, \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}.$$

2. Let  $f = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{R}, x - 2y = 4\}$ .

(a) Find  $f(10)$ .

$$\begin{aligned} 10 - 2y &= 4 \\ 2y &= 6 \\ y &= 3 \\ f(10) &= 3 \end{aligned}$$

(b) Clearly explain why  $f$  is not onto  $\mathbb{R}$ .

There is no  $x \in \mathbb{Z}$  such that, for example,  $f(x) = \frac{1}{3}$

(Some of you thought that  $\mathbb{R}$  was the set of rational numbers.

$\mathbb{Q}$  - for quotient - is the set of rational numbers.  $\mathbb{R}$  is the set of real numbers.)

3. Let  $f : \mathbb{Z}^{\text{odd}} \rightarrow \mathbb{Z}$  be given by  $f(x) = \frac{x-1}{2}$ .

(a) Prove that  $f$  is one-to-one.

Proof: Let  $c, d \in \mathbb{Z}^{\text{odd}}$  such that  $f(c) = f(d)$ .  
Then  $\frac{c-1}{2} = \frac{d-1}{2}$ .  
 $c - 1 = d - 1$ .  
 $c = d$ .

(b) Prove that  $f$  is onto.

Proof: Let  $y \in \mathbb{Z}$ .  
Let  $x = 2y + 1$ .  
Then  $x \in \mathbb{Z}^{\text{odd}}$  by the definition of odd.  
And  $f(x) = f(2y + 1) = \frac{2y+1-1}{2} = \frac{2y}{2} = y$ .

4. Calculations:

(a) Find  $\gcd(182, 140)$ .

$$182 = 140(1) + (42)$$

$$140 = 42(3) + 14$$

$$42 = 14(3) + 0$$

$$\text{So } \gcd(182, 140) = 14.$$

(b) Find  $x$  so that  $x \text{div} 6 = 3$  and  $x \text{mod} 6 = 2$ .

$$x = 3(6) + 2 = 20.$$

5. Let  $f = \{(3, 8), (4, 6), (5, 7), (6, 9)\}$  and  $g = \{(2, 3), (3, 5), (4, 5)\}$ . Find each of the following:

(a)  $f^{-1}(9)$

6

(b)  $f \circ g(3)$ .

7

6. Give an example of each of the following:

(a) A function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  that is not one-to-one.

$$f(x) = x^2$$

(b) An integer that is relatively prime to 12.7

7. (a) Disprove the following by demonstrating a counterexample: If  $a, b, n \in \mathbb{Z}$  such that  $a \bmod n = 1$  then  $ab \bmod n = 1$ .

$$n = 2, a = 3, b = 4.$$

(b) Prove: If  $a, b, n \in \mathbb{Z}$  such that  $a \bmod n = 1$  and  $b \bmod n = 1$  then  $ab \bmod n = 1$

Proof: Hyp:  $a \bmod n = 1, b \bmod n = 1$

$\exists q, s \in \mathbb{Z}$  so that  $a = nq + 1$  and  $b = ns + 1$ .

So  $ab = (nq + 1)(ns + 1) = n^2qs + nq + ns + 1$

Thus  $ab = n(nqs + q + s) + 1$  and  $nqs + q + s \in \mathbb{Z}$  by App. D.

$\therefore ab \bmod n = 1$ .