

1. Prove by Mathematical Induction: $\forall n \in \mathbb{N}, \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$.

Proof: Clearly $\frac{1}{1 \cdot 3} = \frac{0+1}{2(0)+3}$, so the equation holds for $n = 0$.

Suppose that the equation holds for some $k \in \mathbb{N}$.

Then $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$.

$$\begin{aligned} \text{So } \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2(k+1)+1)(2(k+1)+3)} &= \frac{k+1}{2k+3} + \frac{1}{(2(k+1)+1)(2(k+1)+3)} \\ &= \frac{k+1}{2k+3} + \frac{1}{(2k+3)(2k+5)} \\ &= \frac{(k+1)(2k+5)+1}{(2k+3)(2k+5)} \\ &= \frac{2k^2+7k+6}{(2k+3)(2k+5)} \\ &= \frac{(2k+3)(k+2)}{(2k+3)(2k+5)} \\ &= \frac{k+2}{2k+5} \\ &= \frac{(k+1)+1}{2(k+1)+3} \end{aligned}$$

Thus, the equation holds for $k + 1$.

$$\therefore \forall n \in \mathbb{N}, \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}.$$

2. Let $f = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{R}, x - 2y = 4\}$.

(a) Find $f(10)$.

$$\begin{aligned} 10 - 2y &= 4 \\ 2y &= 6 \\ y &= 3 \\ f(10) &= 3 \end{aligned}$$

(b) Clearly explain why f is not onto \mathbb{R} .

There is no $x \in \mathbb{Z}$ such that, for example, $f(x) = \frac{1}{3}$

(Some of you thought that \mathbb{R} was the set of rational numbers.

\mathbb{Q} - for quotient - is the set of rational numbers. \mathbb{R} is the set of real numbers.)

3. Let $f : \mathbb{Z}^{\text{odd}} \rightarrow \mathbb{Z}$ be given by $f(x) = \frac{x-1}{2}$.

(a) Prove that f is one-to-one.

Proof: Let $c, d \in \mathbb{Z}^{\text{odd}}$ such that $f(c) = f(d)$.
Then $\frac{c-1}{2} = \frac{d-1}{2}$.
 $c - 1 = d - 1$.
 $c = d$.

(b) Prove that f is onto.

Proof: Let $y \in \mathbb{Z}$.
Let $x = 2y + 1$.
Then $x \in \mathbb{Z}^{\text{odd}}$ by the definition of odd.
And $f(x) = f(2y + 1) = \frac{2y+1-1}{2} = \frac{2y}{2} = y$.

4. Calculations:

(a) Find $\gcd(182, 140)$.

$$182 = 140(1) + (42)$$

$$140 = 42(3) + 14$$

$$42 = 14(3) + 0$$

$$\text{So } \gcd(182, 140) = 14.$$

(b) Find x so that $x \text{div} 6 = 3$ and $x \text{mod} 6 = 2$.

$$x = 3(6) + 2 = 20.$$

5. Let $f = \{(3, 8), (4, 6), (5, 7), (6, 9)\}$ and $g = \{(2, 3), (3, 5), (4, 5)\}$. Find each of the following:

(a) $f^{-1}(9)$

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(b) $f \circ g(3)$.

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6. Give an example of each of the following:

(a) A function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ that is not one-to-one.

$$f(x) = x^2$$

(b) An integer that is relatively prime to 12.7

7. (a) Disprove the following by demonstrating a counterexample: If $a, b, n \in \mathbb{Z}$ such that $a \bmod n = 1$ then $ab \bmod n = 1$.

$$n = 2, a = 3, b = 4.$$

(b) Prove: If $a, b, n \in \mathbb{Z}$ such that $a \bmod n = 1$ and $b \bmod n = 1$ then $ab \bmod n = 1$

Proof: Hyp: $a \bmod n = 1, b \bmod n = 1$

$\exists q, s \in \mathbb{Z}$ so that $a = nq + 1$ and $b = ns + 1$.

So $ab = (nq + 1)(ns + 1) = n^2qs + nq + ns + 1$

Thus $ab = n(nqs + q + s) + 1$ and $nqs + q + s \in \mathbb{Z}$ by App. D.

$\therefore ab \bmod n = 1$.