2 Exercises

2.1. Please determine which of the following are true and which are false; use Definition 2.2 to explain your answers.

a. $3|100$.

b. $3|99$.

c. $-3|3$.

d. $-5|-5$.

e. $-2|-7$.

f. $0|4$.

g. $4|0$.

h. $0|0$.

2.2. Here is a possible alternative to Definition 2.2: We say that a is *divisible* by b provided $\frac{a}{b}$ is an integer. Explain why this alternative definition is different from Definition 2.2.

Here, *different* means that Definition 2.2 and the alternative definition specify *different concepts*. So, to answer this question, you should find integers a and b such that a is divisible by b according to one definition, but a is not divisible by b according to the other definition.

2.3. None of the following numbers is prime. Explain why they fail to satisfy Definition 2.5. Which of these numbers is composite?

a. 21.

b. 0.

c. π.

d. $\frac{1}{2}$.

e. -2.

f. -1.

2.4. The *natural numbers* are the nonnegative integers; that is,

The symbol $\mathbb{N}$ stands for the natural numbers.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Use the concept of natural numbers to create definitions for the following relations about integers: *less than* ($<$), *less than or equal to* ($\leq$), *greater than* ($>$), and *greater than or equal to* ($\geq$).

Note: Many authors define the natural numbers to be just the positive integers; for them, zero is not a natural number. To me, this seems unnatural ☹. The concepts *positive integers* and *nonnegative integers* are unambiguous and universally recognized among mathematicians. The term *natural number*, however, is not 100% standardized.

2.5. A *rational number* is a number formed by dividing two integers a/b where $b \neq 0$. The set of all rational numbers is denoted $\mathbb{Q}$.

The symbol $\mathbb{Q}$ stands for the rational numbers.

Explain why every integer is a rational number, but not all rational numbers are integers.

2.6. Define what it means for an integer to be a *perfect square*. For example, the integers 0, 1, 4, 9, and 16 are perfect squares. Your definition should begin

> An integer x is called a *perfect square* provided....

2.7. This problem involves basic geometry. Suppose the concept of distance between points in the plane is already defined. Write a careful definition for one point to be *between* two other points. Your definition should begin

> Suppose A, B, C are points in the plane. We say that C is *between* A and B provided....

Note: Since you are crafting this definition, you have a bit of flexibility. Consider the possibility that the point C might be the same as the point A or

B, or even that A and B might be the same point. Personally, if A and C were the same point, I would say that C is between A and B (regardless of where B may lie), but you may choose to design your definition to exclude this possibility. Whichever way you decide is fine, but be sure your definition does what you intend.

Note further: You do not need the concept of collinearity to define *between*. Once you have defined *between*, please use the notion of between to define what it means for three points to be collinear. Your definition should begin

> Suppose A, B, C are points in the plane. We say that they are collinear provided. . . .

Note even further: Now if, say, A and B are the same point, you certainly want your definition to imply that A, B, and C are collinear.

2.8. Discrete mathematicians especially enjoy *counting problems*: problems that ask *how many*. Here we consider the question: How many positive divisors does a number have? For example, 6 has four positive divisors: 1, 2, 3, and 6.

How many positive divisors does each of the following have?

a. 8.
b. 32.
c. 2^n where n is a positive integer.
d. 10.
e. 100.
f. 1,000,000.
g. 10^n where n is a positive integer.
h. $30 = 2 \times 3 \times 5$.
i. $42 = 2 \times 3 \times 7$. (Why do 30 and 42 have the same number of positive divisors?)
j. $2310 = 2 \times 3 \times 5 \times 7 \times 11$.
k. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$.
l. 0.

2.9. An integer n is called *perfect* provided it equals the sum of all its divisors that are both positive and less than n. For example, 28 is perfect because the positive divisors of 28 are 1, 2, 4, 7, 14, and 28. Note that $1+2+4+7+14 = 28$.

a. There is a perfect number smaller than 28. Find it.
b. Write a computer program to find the next perfect number after 28.

2.10. *At a Little League game there are three umpires. One is an engineer, one is a physicist, and one is a mathematician. There is a close play at home plate, but all three umpires agree the runner is out.*

Furious, the father of the runner screams at the umpires, "Why did you call her out?!"

The engineer replies, "She's out because I call them as they are."
The physicist replies, "She's out because I call them as I see them."
The mathematician replies, "She's out because I called her out."
Explain the mathematician's point of view.