element! What does the formula give? Equation (2) says the number of such lists is

$$4 \times 3 \times 2 \times 1 \times 0 \times -1$$

which equals 0. However, the reasoning behind the formula breaks down. Although it is true that there are 4, 3, 2, 1, and 0 choices for positions one through five, it does not make any sense to say there are -1 choices for the last position! Formula (2) gives the correct answer, but the reasoning used to arrive there needs to be rechecked.

In this paragraph, we use Exercise 4.12: If $a, b \in \mathbb{Z}$, then $a < b \iff a \le b - 1$.

If the number of elements from which we select entries in the list, n, is less than the length of the list, k, no repetition-free list is possible. But since $n < k$, we know that $n - k < 0$ and so $n - k + 1 < 1$. Since $n - k + 1$ is an integer, we know that $n - k + 1 \le 0$. Therefore, in the product $n \times (n-1) \times \cdots \times (n-k+1)$, we know that at least one of the factors is zero. Therefore the whole expression evaluates to zero, which is what we wanted!

On the other hand, if $n \ge k$, our reasoning makes sense (all the numbers are positive), and the formula in (2) gives the correct answer.

Because the expression $n(n-1)(n-2)\cdots(n-k+1)$ occurs fairly often, there is a special notation for it. The notation is

The special notation for $n(n-1)\cdots(n-k+1)$ is $(n)_k$. An alternative notation, still in use on some calculators, is ${}_nP_k$.

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1).$$

This notation is called *falling factorial*. We summarize our results on lists with or without repetition concisely using this notation.

Theorem 7.6 The number of lists of length k whose elements are chosen from a pool of n possible elements

$$= \begin{cases} n^k & \text{if repetitions are permitted} \\ (n)_k & \text{if repetitions are forbidden.} \end{cases}$$

I do not recommend that you memorize this result because it is too easy to get confused between the meanings of n and k. Rather, rederive it in your mind when you need it. Imagine the k boxes written out in front of you, put the appropriate numbers in the boxes, and multiply.

Recap

This section deals with counting lists of objects. The central tool is the Multiplication Principle. A general formula is developed for counting length-k lists of elements selected from a pool of n elements either with or without repetitions.

7 Exercises

7.1. A *bit string* is a list of 0s and 1s. How many length-k bit strings can be made?

7.2. Airports have names, but they also have three-letter codes. For example, the airport serving Baltimore is BWI, and the code YYY is for the airport in Mont Joli, Québec, Canada. How many different airport codes are possible?

7.3. A car's ventilation system has various controls. The fan control has four settings: off, low, medium, and high. The air stream can be set to come out at the floor, through the vents, or through the defroster. The air conditioning button can be either on or off. The temperature control can be set to cold, cool, warm, or hot. And finally, the recirculate button can be either on or off.

In how many different ways can these various controls be set?

Note: Several of these settings result in the same effect since nothing happens if the fan control is off. However, the problem asks for the number of different settings of the controls, not the number of different ventilation effects possible.

7.4. My compact disc player has space for 5 CDs; there are five trays numbered 1 through 5 into which I load the CDs. I own 100 CDs.

a. In how many ways can the CD player be loaded if all five trays are filled with CDs?

b. In how many ways can the CD player be loaded if only one CD is placed in the machine?

7.5. You own three different rings. You wear all three rings, but no two of the rings are on the same finger, nor are any of them on your thumbs. In how many ways can you wear your rings? (Assume any ring will fit on any finger.)

7.6. In how many ways can a black rook and a white rook be placed on different squares of a chess board such that neither is attacking the other? (In other words, they cannot be in the same row or the same column of the chess board. A standard chess board is 8×8.)

7.7. License plates in a certain state consist of six characters: The first three characters are uppercase letters (A–Z), and the last three characters are digits (0–9).

a. How many license plates are possible?

b. How many license plates are possible if no character may be repeated on the same plate?

7.8. A telephone number (in the United States and Canada) is a ten-digit number whose first digit cannot be a 0 or a 1. How many telephone numbers are possible?

7.9. A U.S. Social Security number is a nine-digit number. The first digit(s) may be 0.

a. How many Social Security numbers are available?

b. How many of these are even?

c. How many have all of their digits even?

d. How many read the same backward and forward (e.g., 122979221)?

e. How many have none of their digits equal to 8?

f. How many have at least one digit equal to 8?

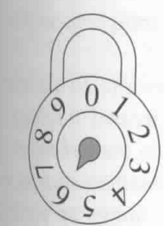
g. How many have exactly one 8?

The word *character* means a letter or a digit.

7.10. A computer operating system allows files to be named using any combination of uppercase letters (A–Z) and digits (0–9), but the number of characters in the file name is at most eight (and there has to be at least one character in the file name). For example, `X23`, `W`, `4AA`, and `ABCD1234` are valid file names, but `W-23` and `WONDERFUL` are not valid (the first has an improper character, and the second is too long).

How many different file names are possible in this system?

7.11. How many five-digit numbers are there that do not have two consecutive digits the same? For example, you would count 12104 and 12397 but not 6321 (it is not five digits) or 43356 (it has two consecutive 3s).
Note: The first digit may not be a zero.



7.12. A padlock has the digits 0 through 9 arranged in a circle on its face. A combination for this padlock is four digits long. Because of the internal mechanics of the lock, no pair of consecutive numbers in the combination can be the same or one place apart on the face. For example 0-2-7-1 is a valid combination, but neither 0-4-4-7 (repeated digit 4) nor 3-0-9-5 (adjacent digits 0-9) are permitted. How many combinations are possible?

7.13. A bookshelf contains 20 books. In how many different orders can these books be arranged on the shelf?

7.14. A class contains ten boys and ten girls. In how many different ways can they stand in a line if they must alternate in gender (no two boys and no two girls are standing next to one another)?

7.15. Four cards are drawn from a standard deck of 52 cards. In how many ways can this be done if the cards are all of different values (e.g., no two 5s or two jacks) and all of different suits? (For this problem, the order in which the cards are drawn matters, so drawing A♤-K♡-3♢-6♧ is not the same as drawing 6♧-K♡-3♢-A♤ even though the same cards are selected.)

8 Factorial

In Section 7, we counted lists of elements of various lengths in which we were either allowed or forbidden to repeat elements. A special case of this problem is to count the number of length-n lists chosen from a pool of n objects in which repetition is forbidden. In other words, we want to arrange n objects into a list, using each object exactly once. By Theorem 7.6, the number of such lists is

$$(n)_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots(1).$$

The quantity $(n)_n$ occurs frequently in mathematics and has a special name and notation; it is called *n factorial* and is written $n!$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Two special cases of the factorial function require special attention.

First, let us consider $1!$. This is the result of multiplying all the numbers starting from 1 all the way down to, well, 1. The answer is 1. Just in case this isn't clear, let's return to the list-counting application. In how many ways can we make a length-1 list where there is only one possible element to fill the first (and only!) position? Obviously, there is only one possible list. So $1! = 1$.

The other special case is $0!$.