Section 10 - Quantifiers

In everyday usage, when we use a collective noun (people, Americans, cats, etc.), sometimes we mean all and sometimes we just mean most. In math, we always mean all unless we explicitly state otherwise. So

"Integers are not zero" is false even though it is true for all integers but one.

The following are all true and identical in meaning:

"There exists an integer that is not zero", "Some integers are not zero", "at least one integer is not zero".

When we want to express how many we mean, we can use phrases such as "for all" if we really mean all, or "there exists" if we mean at least one, but not necessarily all. The symbol for "for all" is $\forall$ and the symbol for "there exists" is $\exists$. So, $\forall x \in \mathbb{Z}, x \neq 0$ is false, and $\exists x \in \mathbb{Z}, x \neq 0$ is true. $\forall$ statements are called universal statements. $\exists$ statements are called existential statements.

Some of our earlier if-then statements can be reformulated using this notation. So "If x is even then $x + 9$ is odd" can be written as "$\forall x \in \mathbb{Z}^{\text{even}}, x + 9 \in \mathbb{Z}^{\text{odd}}$."

Ex: Rewrite "Every integer is less than or equal to its square."

$$\text{Ans: } \forall x \in \mathbb{Z}, x \leq x^2$$

Combining Quantifiers

Now, consider a combined statement, such as $\forall x \in \mathbb{Q}^+, \exists y \in \mathbb{Z}, xy = 1$. This says "for every positive rational number, there is an integer, so that the product of the two numbers is 1". This is true.

Now, consider $\exists x \in \mathbb{Q}^+, \forall y \in \mathbb{Z}, xy = 1$. This says "There is a positive rational number, so that for every integer, the product of the two numbers is 1". This is false. In order for it to be true, we would have to have a single rational number that multiplied with every integer to give one.

$\forall x \in \mathbb{Q}^+, \forall y \in \mathbb{Z}, xy = 1$ is also false.
$\exists x \in \mathbb{Q}^+, \exists y \in \mathbb{Z}, xy = 1$ is true.

Now, consider
$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = 0.$ is true. $(x = 0)$.
$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 0.$ is true. $(x = 0$ or $y = 0)$.
$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = 0.$ is falso. (xy is not always 0).
$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 0.$ is true. $(y = 0)$.

Negating Quantified Statements

Consider "There is an integer that is less than itself."

$$\exists x \in \mathbb{Z}, x < x$$

This is false. The negation of it will be true.

The negation is "There is no integer that is less than itself".

We could express this as:

$$\nexists x \in \mathbb{Z}, x < x$$

but, this is clumsy. Instead we use the fact that to negate an existential statement, we can use a universal statement.

$$\neg (\exists x, \text{claim}) = \forall x, \neg \text{ claim}.$$

So, we can rewrite the above as:

$$\forall x \in \mathbb{Z}, x \geq x.$$

Be careful when doing this to make sure that you have the opposite of the claim correct. For example, we have not yet shown that all integers are even or odd but not both. So to negate

$$\exists x \in \mathbb{Z}, 2x \text{ is even}$$

we can't use

$$\forall x \in \mathbb{Z}, 2x \text{ is odd}$$

we have to use

$$\forall x \in \mathbb{Z}, 2x \text{ is not even}$$

To negate a universal statement, we do the same thing. The negation of

$$\forall x \in \mathbb{Z}, \frac{1}{x} \in \mathbb{Q}.$$

is

$$\exists x \in \mathbb{Z}, \frac{1}{x} \notin \mathbb{Q}.$$

Now, for combined statements, we just keep applying these rules. The negation of

$$\exists x \in \mathbb{Q}^+, \forall y \in \mathbb{Z}, xy = 1$$

is

$$\begin{aligned} & \neg \left(\exists x \in \mathbb{Q}^+, \forall y \in \mathbb{Z}, xy = 1\right) \\ \forall x & \in \mathbb{Q}^+, \neg (\forall y \in \mathbb{Z}, xy = 1) \\ \forall x & \in \mathbb{Q}^+, \exists y \in \mathbb{Z}, \neg (xy = 1) \\ \forall x & \in \mathbb{Q}^+, \exists y \in \mathbb{Z}, xy \neq 1. \end{aligned}$$

Homework: Section 10, P. 63 #1,2,4,5