Section 19 - Proof by Contradiction and Proof by Contrapositive

Recall that 'If A then B' is equivalent to 'If $\neg B$, then $\neg A$'. The second statement is the contrapositive of the first (and vice versa).

Ex: Prove: If $5 \nmid x$, then $10 \nmid x$.
Now, we don't want to set this proof up the usual way. If we do, then we would start with $5 \nmid x$. But what could we do with that? There is not an integer a so that $5a = x$. Bleccchhh! So, instead, we use the contrapositive: If $10 \mid x$, then $5 \mid x$. This we know how to prove. Since it is equivalent to the original, we have proved the original.

Now, sometimes we want to use proof by contradiction. We have already seen this before, but let's set up a formal template for it.

If we want to prove 'If A then B', we can use the following template (note that 'for the sake of contradiction' is short for 'in the hopes of arriving at a logical contradiction when following this train of thought':

Hyp	:	A
		Suppose, for the sake of contradiction, $\neg B$.
		$\vdots$
		Some contradiction. (That is, something that can't be true.)
		This isn't true, because ....
		So we have arrived at a contradiction.
Conc	:	B

Ex: Prove: If $5 \nmid x$, then $10 \nmid x$.

Proof :	Hyp:	$5 \nmid x$.
		Suppose, for the sake of contradiction, $10 \mid x$.
		Then $x = 10a$ for some $a \in \mathbb{Z}$.
		So $x = 5(2a)$ and $2a \in \mathbb{Z}$.
		Thus $5 \mid x$.
		This isn't true, because our hypothesis is that $5 \nmid x$.
		So we have arrived at a contradiction.
Conc	:	$10 \nmid x$.

Most proofs that can be done by contradiction can be done by contrapositive, but not all.

Ex: Prove: If A is a set, then $A \cap \emptyset = \emptyset$.
The contrapositive would be: If $A \cap \emptyset \neq \emptyset$, then A is not a set. So this is not a good

approach. Let's try proof by contradiction.

Proof:	Hyp:	A is a set.
		Suppose, for the sake of contradiction, $A \cap \emptyset \neq \emptyset$.
		Then there exists at least one element in $A \cap \emptyset$.
		Let $x \in A \cap \emptyset$.
		Then $x \in A$ and $x \in \emptyset$.
		This isn't true, because we can't have $x \in \emptyset$.
		So we have arrived at a contradiction.
	Conc:	$A \cap \emptyset = \emptyset$.

Homework: Section 19, P. 141 #1-6,8,9,11 (you may want to use 'least' in place of 'smallest')