Section 21 - Proof by Induction

First do worksheet.

Mathematical Induction

Prove: $\forall n \in \mathbb{N}$, some equation, inequality, or property.
Proof:

$\vdots$

Thus the equation, inequality, or property holds with $n = 0$.
Suppose the equation, inequality, or property holds with $n = k$ for some $k \in \mathbb{N}$.
Then equation, inequality, or property with k in place of n.

$\vdots$

Equation, inequality, or property with $k+1$ in place of n.
So, the equation, inequality, or property holds with $n = k+1$
$\therefore$ $\forall n \in \mathbb{N}$, the equation, inequality, or property.

Ex: Prove: $\forall n \in \mathbb{N}$, $0 + 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Proof: We can clearly see that $0 = \dfrac{0(0+1)}{2}$ is true.
Thus the equation holds with $n = 0$.
Suppose the equation holds with $n = k$ for some $k \in \mathbb{N}$.
Then $0 + 1 + 2 + \ldots + k = \dfrac{k(k+1)}{2}$
So, adding $k+1$ to both sides, we have:

$$\begin{aligned}
0 + 1 + 2 + \ldots + (k+1) &= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2 + k + 2k + 2}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
0 + 1 + 2 + \ldots + (k+1) &= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

So, the equation holds with $n = k+1$.
$\therefore$ $\forall n \in \mathbb{N}$, $0 + 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Ex: Prove: $\forall n \in \mathbb{N}$ with $n \geq 3$, $2n+1 < 2^n$.

Proof: $2(3)+1 = 7$ and $2^3 = 8$.
So $2(3)+1 < 2^3$.
Thus the inequality holds with $n = 3$.
Suppose the inequality holds with $n = k$ for some $k \in \mathbb{N}$ with $k \geq 3$
Then $2k+1 < 2^k$.
So $2k+1+2 < 2^k+2$
$2k+3 < 2^k+2^k$
$2k+3 < 2\left(2^k\right)$
$2(k+1)+1 < 2^{(k+1)}$.
So, the inequality holds with $n = k+1$.
$\therefore$ $\forall n \in \mathbb{N}$ with $n \geq 3$, $2n+1 < 2^n$.

Ex: Prove: $\forall n \in \mathbb{N}$, $3 \mid n^3 - 7n + 3$.

Proof: $0^3 - 7(0) + 3 = 3$, which is clearly divisible by 3.
Thus the property holds with $n = 0$.
Suppose the property holds with $n = k$ for some $k \in \mathbb{N}$.
Then $3 \mid k^3 - 7k + 3$
$k^3 - 7k + 3 = 3m$ for some $m \in \mathbb{Z}$.
$k^3 - 7k + 3 + 3k^2 + 3k - 6 = 3m + 3k^2 + 3k - 6$
$k^3 - 7k + 3 + 3k^2 + 3k - 6 = 3m + 3k^2 + 3k - 6$
$k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 = 3m + 3k^2 + 3k - 6$
$(k+1)^3 - 7(k+1) + 3 = 3(m + k^2 + k - 2)$ and $m + k^2 + k - 2 \in \mathbb{Z}$.
$3 \mid (k+1)^3 - 7(k+1) + 3$
So, the property holds with $n = k+1$.
$\therefore$ $\forall n \in \mathbb{N}$, $3 \mid n^3 - 7n + 3$.

Homework: Section 21, P. 168 #1,3,4, and Propositions 21.4,5,6,7 in the section.