Section 23 - Functions

Recall a relation from a set A to a set B is a subset of $A \times B$. We write ${}_xR_y$ to mean that $(x, y) \in R$.

Def: A relation f is a *function* if there do not exist $a \in A, b, c \in B$ such that $b \neq c$ and $(a, b), (a, c) \in R$

Ex: Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$. Then $\{(1, 6), (2, 9), (3, 9)\}$ is a function, but $\{(1, 6), (2, 9), (2, 7)\}$ is not.

If $(a, b) \in f$, we could ${}_af_b$ as before, but instead when f is a function, we use a different notation. We write $f(a) = b$.

Ex: Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$ and $f = \{(1, 6), (2, 9), (3, 9)\}$. Then $f(1) = 6, f(2) = 9, f(3) = 9$.

Ex: Let $A = Z$ and $B = Z$ and $f = \{(x, y) : y = 5x\}$. Then $f(1) = 5, f(-4) = -20$, etc.

Def: Let f be a function. The *domain* of f is the set $\{x : (x, y) \in f\}$. We denote the domain by dom f.

Def: Let f be a function. The *image* of f is the set $\{y : (x, y) \in f\}$. We denote the image by im f.

Notation: We write $f : A \to B$ if f is a function and dom $f = A$ and im $f \subseteq B$.

Ex: Let $f : Z^+ \to R$ given by $f(x) = \sqrt{x}$.

Pictorially, we can represent functions on discrete sets as before (for relations).

Ex: Let $f : \{1, 2, 3, 4\} \to \{3, 4, 5\}$ be given by $f = \{(1, 3), (2, 5), (3, 3), (4, 5)\}$.

Ex: Let $f : Z \to Z$ be given by $f(x) = x + 2$.

Pictorially, a function will have a single arrow coming from each element in the first set.

Def: Let $f : A \to B$. Then f is *onto* if im $f = B$.

Here we must be careful. Note that $f : Z \to Z$, given by $f(x) = x + 2$ is onto, and $f : R \to R$, given by $f(x) = x + 2$ is onto, but $f : Z \to R$, given by $f(x) = x + 2$ is not.

Pictorially, onto functions will have at least one arrow going to each element in the second set.

Def: Let $f : A \to B$. Then f is *one-to-one* if $f(c) = f(d)$ implies $c = d$.

Exs: $f : Z \to Z$ given by $f(x) = x + 2$ is one-to-one. $f : \{1, 2, 3, 4\} \to \{3, 4, 5\}$ given by $f = \{(1, 3), (2, 5), (3, 3), (4, 5)\}$ is not one-to-one.

Again, we must be careful. $f : Z \to Z$ given by $f(x) = x^2$ is not one-to-one, but $f : Z^+ \to Z$ given by $f(x) = x^2$ is.

Pictorially, one-to-one functions will have no more than one arrow going to each element in the second set.

Def: A function that is one-to-one and onto is called a *bijection.*

Every function has an inverse. We define f^{-1} as before for relations.

Ex: Let $f : \{1,2,3,4\} \to \{3,4,5\}$ be given by $f = \{(1,3),(2,5),(3,3),(4,5)\}$. Then $f^{-1} = \{(3,1),(5,2),(3,3),(5,4)\}$.

Ex: Let $f : Z \to Z$ be given by $f(x) = x + 2$. Then $f = \{(x,y) : y = x + 2\}$ and $f^{-1} = \{(x,y) : y = x - 2\}$.

Pictorially, we get f^{-1} by reversing the arrows in f.

Theorem: Let $f : A \to B$. Then $f^{-1} : B \to A$ is a function if and only if f is a bijection.

Homework: Section23, P. 203 #1(a-i),2,3,5,6,12