Section 25 - Composing Functions

Def: Let A, B, C be sets and let $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ is given by $g \circ f(x) = g(f(x))$.

Ex: Let $f : \mathbb{Z} \to \mathbb{Z}, g : \mathbb{Z} \to \mathbb{R}$, be given by $f(x) = 5x + 1$, $g(x) = \sqrt[3]{x-2}$. Then $g \circ f : \mathbb{Z} \to \mathbb{R}$ and is given by $g \circ f(x) = \sqrt[3]{(5x+1)-2} = \sqrt[3]{5x-1}$

Ex: Let $f : \mathbb{Z}^+ \to \mathbb{R}, g : \mathbb{R} \to \{-1, 1\}$, be given by $f(x) = \sqrt{x}$, $g(x) = \begin{cases} -1 & \text{if } x < 4 \\ 1 & \text{if } x \geq 4 \end{cases}$.

Then $g \circ f : \mathbb{Z}^+ \to \{-1, 1\}$ and is given by $g \circ f(x) = \begin{cases} -1 & \text{if } \sqrt{x} < 4 \\ 1 & \text{if } \sqrt{x} \geq 4 \end{cases}$

Pictorially, B is a stopover on the way from A to C.

Note: Function composition is not a commutative relation (that is, $f \circ g$ is not usually the same as $g \circ f$, even when their domains are the same.) If is an associative operation (that is, $h \circ (g \circ f) = (h \circ g) \circ f$). We can't prove this yet, though because we have not yet defined what it means for two functions to be equal. We will do that next time.

Def: Let A be a set. The *identity function* on A is the function $id_A : A \to A$ given by $\text{id}_A(x) = x$.

Let's prove this is a bijection.

Proof: Hyp: Let $c, d \in A$ with $\text{id}_A(c) = \text{id}_A(d)$.
Conc: $c = d$.

So, id_A is one-to-one.

Proof: Hyp: Let $y \in A$.
Let $x = y$.
Then $x \in A$ because $y \in A$.
And $f(x) = x = y$.
Conc: $\exists x \in A$ such that $f(x) = y$.

So, id_A is onto. Thus, it is a bijection.

Homework: Section25, P. 215 #1,8,9,10,11