Section 25 - Proving two functions are equal.

Def: Let f and g be functions. Then we write $f = g$ if dom f =dom$g$ and $f(x) = g(x)$ for all x in the domain.

Here is the template for proving two functions are equal.

Proof: Hyp: f and g are functions.
$\vdots$
dom$f$ =dom g
Let $x \in$dom$f$. Then $x \in$dom$g$ also.
$\vdots$
Thus $f(x) = g(x)$.
Conc: $f = g$.

Ex (#7 on P. 215): Let $f : A \to B, g : B \to A$. Prove: If $g \circ f$ =id$_A$ and $f \circ g$ =id$_B$ then $f^{-1} : B \to A$ and $g = f^{-1}$.

We need to prove 3 things. 1) f is onto. 2) f is one-to-one. 3) $g = f^{-1}$.

Proving f is one-to-one:

Proof: Hyp: Let $c, d \in A$ with $f(c) = f(d)$.
Since $f(c), f(d) \in B$, $g(f(c))$ and $g(f(d))$ are defined.
Further, $g(f(c)) = g(f(d))$
So, id$_A(c)$ =id$_A(d)$.
Conc: $c = d$.

Proving f is onto:

Proof: Hyp: Let $y \in B$.
Let $x = g(y)$.
Then $x \in A$ because $g : B \to A$.
And $f(x) = f(g(y))$ =id$_B(y) = y$.
Conc: $\exists x \in A$ such that $f(x) = y$.

Proving $g = f^{-1}$:

Proof: Hyp: $f^{-1} : B \to A$ and $g : B \to A$ are functions.
dom$f = B$ =dom g
Let $x \in B$.
Then let $a = f^{-1}(x)$.
By definition of f^{-1}, $f(a) = x$.
Thus $g(x) = g(f(a))$ =id$_A(a) = a = f^{-1}(x)$
Conc: $f^{-1} = g$.

Homework: Section25, P. 215 #3,5,6,12