Section 4 - If-Then Proofs and Iff Proofs

Ex: Prove that the sum of an odd integer and an even integer is odd.
Ans:

In if-then form: If x is an odd integer and y is an even integer, then $x + y$ is odd.

Proof: Hypothesis: $\boxed{\text{Let } x \text{ be an odd integer and } y \text{ be an even integer.}}$
Then, by the definition of odd, $x = 2a + 1$ for some integer a.
And, by the definition of even, $y = 2b$ for some integer b.
$x + y = 2a + 1 + 2b$
$x + y = 2(a + b) + 1.$
And, $a + b$ is an integer by the closure property for addition of integers.
Conclusion: $\boxed{x + y \text{ is odd.}}$

Or

We assume that x is odd and y is even.
Then there exist some integers a and b so that $x = 2a$ and $y = 2b + 1,$
by the definition of even and odd, respectively.
Since $x + y = 2a + 2b + 1 = 2(a + b) + 1,$ and the closure property for addition of integers guarantees that $a + b$ is an integer, we have $x + y$ odd.

Or

Proof: Hypothesis: $\boxed{\text{Let } x, y \in \mathbb{Z}, \text{ with } x \text{ odd and } y \text{ even.}}$
Then, by the definition of odd, $x = 2a + 1$ for some $a \in \mathbb{Z}$.
And, by the definition of even, $y = 2b$ for some $b \in \mathbb{Z}$.
$x + y = 2a + 1 + 2b$
$x + y = 2(a + b) + 1.$
And, $a + b \in \mathbb{Z}$, since $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.
Conclusion: $\boxed{x + y \text{ is odd.}}$

Ex: Prove that the square of a rational number is rational.number.
Ans:

In if-then form: If q is rational, then q^2 is rational.

Proof: Hypothesis: $\boxed{\text{Let } q \in \mathbb{Q}.}$
Then, $q = \dfrac{a}{b}$ for some $a, b \in \mathbb{Z}$, with $b \neq 0$.
$$q^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$
Since $b \neq 0$, we have $b^2 \neq 0$
And, since $a, b \in \mathbb{Z}$, it follows that $a^2, b^2 \in \mathbb{Z}$
Conclusion: $\boxed{q^2 \in \mathbb{Q}.}$

Ex: Prove: For any integer x, $-x$ is even if and only if x is even.
Ans:

In if-then form: 1) If $-x$ is even, then x is even and 2) If x is even then $-x$ is even.

Proof: Let x be an integer.
($\Longrightarrow$) Hypothesis: $\boxed{-x \text{ is even.}}$
$-x = 2a$ for some $a \in \mathbb{Z}$.
$x = -2a = 2(-a)$.
Now, $-a \in \mathbb{Z}$ by the inverse property of integers.
Conclusion: $\boxed{x \text{ is even.}}$
($\Longleftarrow$) Hypothesis: $\boxed{x \text{ is even.}}$
$x = 2b$ for some $b \in \mathbb{Z}$.
$-x = -2b = 2(-b)$.
Now, $-b \in \mathbb{Z}$ by the inverse property of integers.
Conclusion: $\boxed{-x \text{ is even.}}$

Homework: Section 4, P. 25 #1,3,5,6,8,10,11,13-15