Section 6 - Boolean Algebra

We are going to revisit the operations and truth tables from section 3, but we are now going to view them as algebraic operations, instead of as logical operations.

Notation: $\wedge$ - and, $\vee$ - or, $\neg$ - not, $\rightarrow$ - implies (or $\Longrightarrow$), $\leftrightarrow$ - iff (or $\Longleftrightarrow$)

for statements we will usually use x, y, z ... instead of A, B, C ... to emphasize the nature of alebraic operations.

Recall the truth table for $x \vee y$

x	y	$x \vee y$
T	T	T
T	F	F
F	T	F
F	F	F

Now, suppose we compound this statement to make $\neg (x \vee y)$

x	y	$x \vee y$	$\neg (x \vee y)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

In this way, we can build up more complex statements and check their truth values.

The order of operations is: parentheses, $\neg$, $\vee$ and $\wedge$ and $\rightarrow$ and $\leftrightarrow$.

Ex: Create the truth table for $\neg x \wedge \neg y$

x	y	$\neg x$	$\neg y$	$\neg x \wedge \neg y$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note, these truth tables demonstrate that $\neg (x \vee y)$ is logically equivalent to $\neg x \wedge \neg y$, that is, if the same truth values are given for the input variables x and y, then the two statements will have the same truth value as each other.

We write $\neg (x \vee y) = \neg x \wedge \neg y$

Theorem 6.2 (all statements easily verified by truth tables):

Commutative Properties : $x \wedge y = y \wedge x \qquad x \vee y = y \vee x$

Associative Properties : $(x \wedge y) \wedge z = x \wedge (y \wedge z) \qquad (x \vee y) \vee z = x \vee (y \vee z)$

Identity Elements : $x \wedge \text{TRUE} = x \qquad x \vee \text{FALSE} = x$

Inverse : $\neg (\neg x) = x \qquad x \wedge \neg x = \text{FALSE} \qquad x \vee \neg x = \text{TRUE}$

Distributive Properties : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

DeMorgan's Laws : $\neg (x \vee y) = \neg x \wedge \neg y \qquad \neg (x \wedge y) = \neg x \vee \neg y$

Misc : $x \wedge x = x \qquad x \vee x = x$

We now have two different formal ways to show that statements in Boolean Algebra are equivalent.

Ex: Show that $\neg x \vee (x \wedge y) = \neg x \vee y$.

Method 1:

$$\begin{aligned} \neg x \vee (x \wedge y) &= (\neg x \vee x) \wedge (\neg x \vee y) \\ &= \text{TRUE} \wedge (\neg x \vee y) \\ &= \neg x \vee y \end{aligned}$$

Method 2:

x	y	$\neg x$	$\neg x \vee y$	$x \wedge y$	$\neg x \vee (x \wedge y)$
T	T	F	T	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	F	T

x	y	$\neg x$	$\overline{\neg x \vee y}$	$x \wedge y$	$\overline{\neg x \vee (x \wedge y)}$
T	T	F	T	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	F	T

Homework: Section 6, P. 32 #1,3,6-8,10,11(a,b),12,14,15(a,b,d),19