Math 301 Quiz #7 - Take Home Name: KEY

This quiz is due on Wednesday, April 29, at the end of our class period.

1. Prove by Mathematical Induction: $\forall n \in \mathbb{N},\ 1+3+3^2+\ldots+3^n = \frac{1}{2}\left(3^{n+1}-1\right).$

Proof: We can clearly see that $3^0 = \frac{1}{2}\left(3^{0+1}-1\right)$ is true.
Thus the equation holds with $n=0$.
Suppose the equation holds with $n=k$ for some $k \in \mathbb{N}$.
Then $1+3+3^2+\ldots+3^k = \frac{1}{2}\left(3^{k+1}-1\right)$
So, adding 3^{k+1} to both sides, we have:

$$\begin{aligned} 1+3+3^2+\ldots+3^k+3^{k+1} &= \frac{1}{2}\left(3^{k+1}-1\right)+3^{k+1} \\ &= \frac{1}{2}\left(3^{k+1}-1+2\left(3^{k+1}\right)\right) \\ &= \frac{1}{2}\left(3\left(3^{k+1}\right)-1\right) \\ &= \frac{1}{2}\left(3^{k+2}-1\right) \\ 1+3+3^2+\ldots+3^k+3^{k+1} &= \frac{1}{2}\left(3^{k+1+1}-1\right) \end{aligned}$$

So, the equation holds with $n=k+1$.
$\therefore$ $\forall n \in \mathbb{N},\ 1+3+3^2+\ldots+3^n = \frac{1}{2}\left(3^{n+1}-1\right)$.

2. (a) Carefully explain why $f = \{(x,y) : x,y \in Z, 2x+y>5\}$ is not a function.

$(1,4)$ and $(1,5)$ are both in f. That is, we have a single input assigned more than one output. So f is not a function.

(b) Carefully explain why $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2+1$ is not one-to-one.

$(-1,2)$ and $(1,2)$ are both in f. So f is not one-to-one.

3. Let $f : \mathbb{Z} \to \mathbb{Z}^{\text{even}}$ be given by $f(x) = 2x$. Prove f is a bijection.

Proof: (First, we show that f is one-to-one.)
Let $c,d \in \mathbb{Z}$ with $f(c) = f(d)$.
Then $2c = 2d$.
$c = d$.
$\therefore f$ is one-to-one.
(Now we show that f is onto.)
Let $y \in \mathbb{Z}^{\text{even}}$.
Let $x = \frac{y}{2}$.
Then $x \in \mathbb{Z}$ by the definition of even.
And $f(x) = f\left(\frac{y}{2}\right) = 2\left(\frac{y}{2}\right) = y$.
$\therefore f$ is onto $\mathbb{Z}^{\text{even}}$.
So f is a bijection.