1. (a) For the following differential equation, describe the behavior of y as $t \to \infty$.

$$\frac{dy}{dt} = 2y - 6$$

y diverges from 3.
(That is, if $y(0) > 3$, then $y \to \infty$, and if $y(0) < 3$, then $y \to -\infty$.)

(b) For the following differential equation:

$$y'' - 3y' - 4y = 0$$

Find all possible solutions of the form $y = e^{kx}$.

$$\begin{aligned} k^2 e^{kx} - 3ke^{kx} - 4e^{kx} &= 0 \\ k^2 - 3k - 4 &= 0 \\ (k-4)(k+1) &= 0 \\ k &= 4, -1 \end{aligned}$$

$$y = e^{4x} \text{ or } y = e^{-x}$$

2. Find the appropriate integrating factor for the given differential equation that would allow you to solve (at least implicitly) the differential equation using the techniques of this class. **You do not need to solve the equation, only find the integrating factor!**

(a) $\frac{dy}{dt} + 10ty = t + 3$

$$u(t) = e^{\int 10t\ dt} = e^{5t^2}$$

(b) $(3x+4)\sin y + 4x\cos y \frac{dy}{dx}$

$$\frac{M_y - N_x}{N} = \frac{(3x+4)\cos y - 4\cos y}{4x \cos y} = \frac{3x}{4x} = \frac{3}{4}$$

$$\begin{aligned} \frac{du}{u} &= \frac{3}{4}dx \\ \ln u &= \frac{3}{4}x \\ u &= e^{3x/4} \end{aligned}$$

3. Consider the following autonomous differential equation:

$$\frac{dy}{dt} = (y-1)^2 (y-3) = y^3 - 5y^2 + 7y - 3 = f(y)$$

(a) Sketch the graph of $f(y)$ versus y and indicate all relevant information.

$$\begin{aligned} \text{critical points are at } y &= 1, y = 3. \\ f'(y) &= 3y^2 - 10y + 7 = (3y-7)(y-1) \\ \text{inflection points are at } y &= \frac{7}{3}. \end{aligned}$$

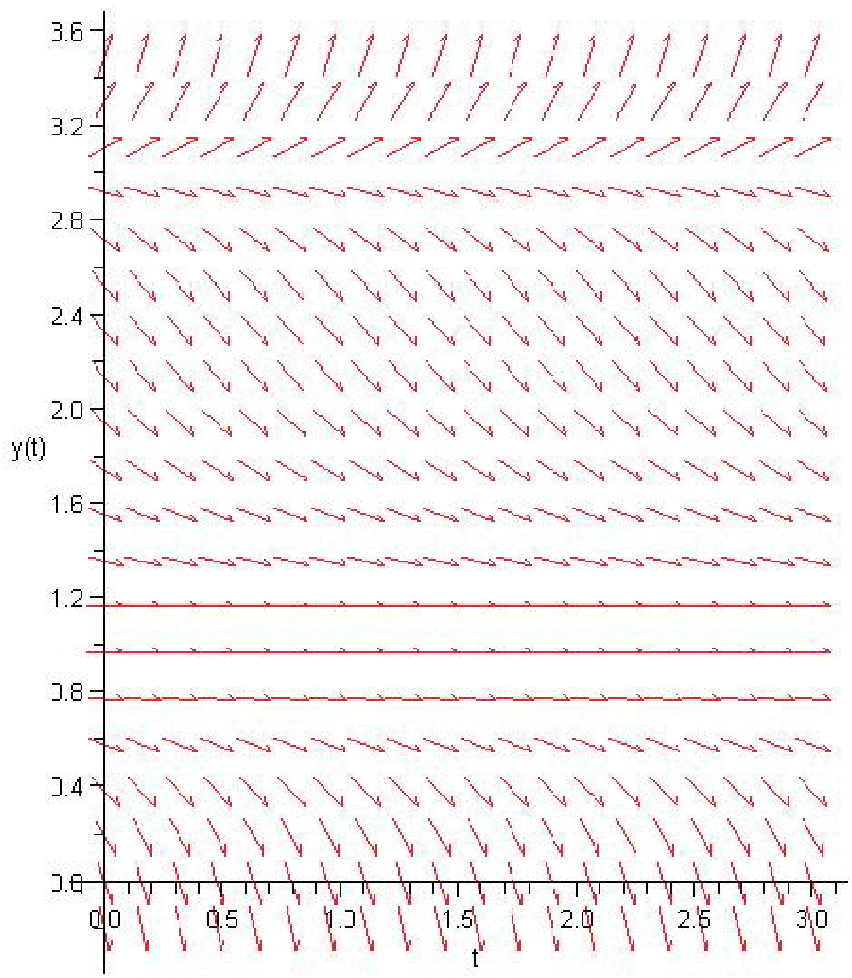
(b) Sketch several graphs of solutions in the ty−plane, making sure to include enough detail so that

i. No type of solution is omitted.

ii. The stability of all equilibrium solutions can be visually determined.

iii. The shape of each solution is clear.

iv. The location of any inflection points is clear.



4. Find the explicit solution to the following initial value problem.

$$\frac{dy}{dx} = e^{-y}\cos x, \qquad y(0) = 2$$

$$\begin{aligned} e^y dy &= \cos x \; dx \\ e^y &= \sin x + C \\ e^2 &= 0 + C \\ C &= e^2 \\ e^y &= \sin x + e^2 \\ y &= \ln\left(\sin x + e^2\right) \end{aligned}$$

5. Suppose that a tank holds 500 liters of a saltwater solution containing 30 grams of salt. At time $t = 0$, a pump system is turned on that allows a saltwater solution with 2 grams of salt per liter to flow into the tank at a rate of 4 liters per minute. The system also causes the solution in the tank to flow out at the same rate. Create an initial value problem to model this situation.

$$\frac{dy}{dt} = (2)(4) - \frac{y}{500}(4)$$

$$\frac{dy}{dt} = 8 - \frac{y}{125}, y(0) = 30$$

6. Find the general (implicit) solution to the following exact differential equation.

$$6xy^5 - 1 + \left(15x^2y^4 + 2\right)\frac{dy}{dx} = 0$$

$$\begin{aligned} \phi(x, y) &= 3x^2y^5 - x + f(y) \\ \phi(x, y) &= 3x^2y^5 + 2y + g(x) \end{aligned}$$

Solution: $3x^2y^5 + 2y - x = C$

7. Miscellaneous

(a) Let $y = \phi(t)$ be the solution to the intial value problem below. Use one step, with stepsize 0.1, of Euler's Method to approximate $\phi(0.1)$

$$\frac{dy}{dt} = \left(t^2 - 2\right)\left(y^2 + t\right), y(0) = 3.$$

$$\begin{aligned} (t_0, y_0) &= (0, 3) \\ m &= \left(t_0^2 - 2\right)\left(y_0^2 + t_0\right) = -18 \end{aligned}$$

$$\begin{aligned} y - y_0 &= m(t - t_0) \\ y_1 - 3 &= -18(0.1) \\ y_1 &= -1.8 + 3 = 1.2 \end{aligned}$$

(b) Consider the following 3 differential equations.

$$\text{DE 1:} \quad \frac{d^2y}{dt^2} - \frac{3}{y} = 5t$$

$$\text{DE 2:} \quad \frac{dy}{dt} = t^2y^2$$

$$\text{DE 3:} \quad \frac{d^2y}{dt^2} = t^3\frac{dy}{dt} - (t+4)\,y + 1$$

i. Which of these are linear?

Only DE3

ii. Which of these are second-order?

DE1 and DE3