1. Solve the initial value problem.

$$y'' + 3y' + 2y = 0, \qquad y(0) = 5, \qquad y'(0) = -3$$

$$\begin{aligned} r^2 + 3r + 2 &= 0 \\ (r+2)(r+1) &= 0 \\ r &= -2, -1 \end{aligned}$$

$$\begin{aligned} y(t) &= c_1 e^{-2t} + c_2 e^{-t} \\ y'(t) &= -2c_1 e^{-2t} - c_2 e^{-t} \end{aligned}$$

$$\begin{aligned} c_1 + c_2 &= 5, -2c_1 - c_2 = -3 \\ c_1 &= -2, c_2 = 7 \end{aligned}$$

$$y(t) = -2e^{-2t} + 7e^{-t}$$

2. (a) Demonstrate that the following set of functions is linearly independent by computing a Wronskian:

$$\left\{te^t, e^{-t}, 1\right\}$$

$$W = \begin{vmatrix} te^t & e^{-t} & 1 \\ e^t + te^t & -e^{-t} & 0 \\ 2e^t + te^t & e^{-t} & 0 \end{vmatrix} = 1 + t + 2 + t = 3 + 2t \neq 0$$

(b) Demonstrate that the following set of functions is linearly dependent without computing a Wronskian:

$$\left\{\sin^2 t, \cos^2 t, t^2, 3\right\}$$

$$3\sin^2 t + 3\cos^2 t + 0t^2 - 3 = 0$$

3. Consider the differential equation

$$t^2 y'' + 2ty' - 2y = 0, \;\; t > 0$$

Use the fact that $y_1(t) = \dfrac{1}{t^2}$ is a solution to this equation and the method of Reduction of Order to find a second linearly independent solution of the form $y_2(t) = v(t) \cdot y_1(t)$.

$$\begin{aligned}
y_2 &= vt^{-2} \\
y_2' &= v't^{-2} - 2vt^{-3} \\
y_2'' &= v''t^{-2} - 2v't^{-3} - 2v't^{-3} + 6vt^{-4} \\
&= v''t^{-2} - 4v't^{-3} + 6vt^{-4}
\end{aligned}$$

$$\begin{aligned}
t^2\left(v''t^{-2} - 4v't^{-3} + 6vt^{-4}\right) + 2t\left(v't^{-2} - 2vt^{-3}\right) - 2\left(vt^{-2}\right) &= 0 \\
v'' - 4v'/t + 6v/t^2 + 2v'/t - 4v/t^2 - 2vt/t^2 &= 0 \\
v'' - 2v'/t &= 0 \\
\frac{v''}{v'} &= \frac{2}{t} \\
\ln v' &= 2\ln t \\
v' &= t^2 \\
v &= \frac{1}{3}t^3 \;\; \text{(or just } t^3 \text{ since equation is linear)}
\end{aligned}$$

$$y_2(t) = t$$

4. Solve the following differential equation using the method of Undetermined Coefficients.

$$y'' - y = te^{3t}$$

$$\begin{aligned}
r^2 - 1 &= 0 \\
r &= \pm 1 \\
y_c(t) &= c_1 e^t + c_2 e^{-t}
\end{aligned}$$

$$\begin{aligned}
Y(t) &= (A + Bt)e^{3t} \\
Y'(t) &= 3(A + Bt)e^{3t} + Be^{3t} \\
Y''(t) &= 9(A + Bt)e^{3t} + 3Be^{3t} + 3Be^{3t} \\
&= 9(A + Bt)e^{3t} + 6Be^{3t}
\end{aligned}$$

$$9(A + Bt)e^{3t} + 6Be^{3t} - (A + Bt)e^{3t} = te^{3t}$$

$$\begin{aligned}
9A + 6B - A &= 0, 9B - B = 1 \\
B &= \frac{1}{8}, A = -\frac{3}{4}B = -\frac{3}{32}
\end{aligned}$$

$$y(t) = c_1 e^t + c_2 e^{-t} - \frac{3}{32}e^{3t} + \frac{1}{8}te^{3t}$$

5. Solve the following differential equation using the Method of Variation of Parameters.

$$y'' - 6y' + 9y = \frac{e^{3t}}{t^3}$$

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \\ r &= 3, 3 \end{aligned}$$

$$y_1(t) = e^{3t}, y_2(t) = te^{3t}$$

$$W = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix} = e^{6t}$$

$$\begin{aligned} Y(t) &= -e^{3t} \int_{t_0}^{t} \frac{se^{3s}\left(\frac{e^{3s}}{s^3}\right)}{e^{6s}} ds + te^{3t} \int_{t_0}^{t} \frac{e^{3s}\left(\frac{e^{3s}}{s^3}\right)}{e^{6s}} ds \\ &= -e^{3t} \int_{t_0}^{t} s^{-2} ds + te^{3t} \int_{t_0}^{t} s^{-3} ds \\ &= -e^{3t}\left(-\frac{1}{s}\Big|_{t_0}^{t}\right) + te^{3t}\left(-\frac{1}{2s^2}\Big|_{t_0}^{t}\right) \\ &= \frac{e^{3t}}{t} - \frac{e^{3t}}{t_0} - \frac{e^{3t}}{2t} + \frac{te^{3t}}{2t_0^2} \\ &= \frac{e^{3t}}{2t} - \frac{e^{3t}}{t_0} + \frac{te^{3t}}{2t_0^2} \end{aligned}$$

Let $t_0 = 1$ and then note that the terms with t_0 are linearly dependent with the fundamental solutions set. So, we can drop them.

$$y(t) = c_1 e^{3t} + c_2 te^{3t} + \frac{e^{3t}}{2t}$$

6. (a) A sixth order homogeneous linear differential equation with constant coefficients has a characteristic equation that factors as follows:

$$(r-2)^4 \left(2r^2 - r + 1\right) = 0$$

Find the general solution $y(t)$.

$$\begin{aligned} 2r^2 - r + 1 &= 0 \\ r &= \frac{1 \pm \sqrt{-7}}{4} = \frac{1}{4} \pm i\frac{\sqrt{7}}{4} \end{aligned}$$

$$y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{t/4} \sin\frac{t\sqrt{7}}{4} + c_6 e^{t/4} \cos\frac{t\sqrt{7}}{4}$$

(a) Use the Method of Undetermined Coefficients to find the form of $y(t)$ so that $y(t)$ is a solution to the following differential equation. (Do not solve for the coefficients!)

$$y''' + y'' = 1 + \sin 2t$$

$$\begin{aligned} r^3 + r^2 &= 0 \\ r^2(r+1) &= 0 \\ y_c(t) &= c_1 + c_2 t + c_3 e^{-t} \end{aligned}$$

$$Y(t) = At^2 + B\cos 2t + C\sin 2t$$

$$y(t) = c_1 + c_2 t + c_3 e^{-t} + At^2 + B\cos 2t + C\sin 2t$$

7. For the following differential equation, let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a series solution.

$$4xy'' + xy' + 3x^2 y = 0$$

(a) Find a recurrence relation for the coefficients a_n.

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

$$\begin{aligned} 4x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 3x^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^{n+2} &= 0 \\ \sum_{m=1}^{\infty} 4(m+1)(m) a_{m+1} x^m + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=2}^{\infty} 3a_{m-2} x^m &= 0 \end{aligned}$$

$$\begin{aligned} m &= 1: \qquad 4(2)(1) a_2 + 1(a_1) = 0 \Longrightarrow a_2 = -\frac{1}{8} a_1 \\ m &\geq 2: \qquad 4(m+1)(m) a_{m+1} + m a_m + 3a_{m-2} = 0 \Longrightarrow a_{m+1} = -\frac{m a_m + 3a_{m-2}}{4(m+1)(m)} \end{aligned}$$

(b) If $a_0 = 1$ and $a_1 = 2$, find a_2 and a_3.

$$\begin{aligned} a_2 &= -\frac{1}{8} a_1 = -\frac{1}{8}(2) = -\frac{1}{4} \\ a_3 &= -\frac{2a_2 + 3a_0}{4(3)(2)} = -\frac{2\left(-\frac{1}{4}\right) + 3(1)}{24} = -\frac{5}{48} \end{aligned}$$