Advanced Calculus, Math 381 **26 October 2005**
Exam 2 **Crawford**
Name: ________________________________

No books. *Show all your work.* Good Luck!

1. (10 pts) Give precise and accurate definitions of the following

(a). A series $\sum a_n$ converging

(b). $\limsup s_n$

2. (10 pts) Give precise and accurate statements of the following (**No proofs necessary**.)

(a). The Bolzano-Weierstrass Theorem

(b). The Integral Test

3. (10 pts) Given $s_n = \dfrac{n^2+2n}{3n+4}$

(a). Use the definition (i.e. M-proof) to show that $\lim s_n = +\infty$

(b). Use only limit laws to show that $\lim s_n = +\infty$

4. (8 pts) Determine whether the following series converges or diverges. If it converges, find the **exact** value of the sum.

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^{2n+2}}$$

5. (16 pts) Determine whether the following series converge or diverge. Show all work and clearly state any tests that you use.

(a). $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

(b). $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

6. (14 pts) Given $a_n = \dfrac{1}{n} \cdot \cos^2\left(\dfrac{n\pi}{2}\right)$

(a). Given an example of a monotone subsequence that is not constant. Clearly state n_k.

(b). Find

$\sup\{s_n \mid n > 1\}$ $\sup\{s_n \mid n > 2\}$ $\sup\{s_n \mid n > 3\}$ $\sup\{s_n \mid n > 4\}$

$\inf\{s_n \mid n > 1\}$ $\inf\{s_n \mid n > 2\}$ $\inf\{s_n \mid n > 3\}$ $\inf\{s_n \mid n > 4\}$

(c). Find $\limsup s_n$ and $\liminf s_n$

7. (16 pts) Prove any ***two*** of the following. [Clearly indicate which two. I will <u>not</u> grade more.]

(a). If s_n is bounded sequences in $\mathbb{R}$ and c is a positive number, then $c \limsup s_n \leq \limsup(cs_n)$

(b). If $\liminf s_n = \limsup s_n = +\infty$, then $\lim s_n = +\infty$.

(c). Let s_n be a sequence of positive numbers. If $\lim s_n = +\infty$, then $\lim \left(\frac{1}{s_n}\right) = 0$.

8. (8 pts) New Proof: Prove **<u>only one</u>** of the following. [Clearly indicate which one. I will <u>not</u> grade both.]

(a). Let $\sum a_n$ be a series of positive terms (i.e. $a_n > 0$) for all n.
If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges. [Hint: Use Cauchy Criterion]

OR

(b). Let $\sum a_n$ be a series of positive terms (i.e. $a_n > 0$) for all n.
If $\sum a_n$ diverges to $+\infty$ and $a_n \leq b_n$ for all n, then $\sum b_n$ diverges to $+\infty$
[Hint: Consider sequence of partial sums]

9. (10 pts) Determine if the following statements are true or false.
<u>Bonus</u>: If the statement is false, give a counterexample or explain why it is impossible.

T F **(a).** If $\lim s_n = +\infty$ and t_n is bounded, then $\lim(s_n t_n) = +\infty$.

T F **(b).** If $\lim s_n = +\infty$ and t_n is bounded, then $\lim(s_n + t_n) = +\infty$.

T F **(c).** If $\lim s_n = +\infty$, then $\exists N$ such that $s_{n+1} \geq s_n$ for all $n > N$.

T F **(d).** If $\liminf s_n = \limsup s_n$, then $\inf s_n = \sup s_n$

T F **(e).** If $\liminf s_n = \limsup s_n$, then $\sup\{s_n \mid n > N\} = \limsup s_n$ for all N

If needed: A series $\sum a_n$ satisfies the <u>Cauchy Criterion</u> if its sequence of partial sums s_k is a Cauchy sequence.

i.e. For each $\epsilon > 0$, there exists an N such that $|s_k - s_m| < \epsilon$ whenever $k, m > N$

or: For each $\epsilon > 0$, there exists an N such that $\left|\sum_{n=m}^{k} a_n\right| < \epsilon$ whenever $k \geq m > N$