MTH 362-01 TEST 1 Solutions
Fall 2008

Please show your work on separate sheets of paper.

1. Define / state
a) a set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ of m-dimensional vectors is linearly independent
b) a matrix A is nonsingular.
c) two systems of linear equations in n unknowns are equivalent
d) Describe/list the three elementary row operations
e) a system of equations is consistent

Solution - see the book

2. Determine whether the vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent or linearly dependent. (Show your work for full credit)

Solution: To be linearly independent, $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} =$ $\boldsymbol{\theta}$ cannot have a nonzero solution. This is equivalent to solving the matrix equation

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, using row reduction techniques:

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \overset{R_2+R_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \overset{-R_2 \& -\frac{1}{2}R_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, $c_1 = c_2 = c_3 = 0$. Hence the vectors are linearly independent.

3.&4 The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ has an inverse.

a) Use echelon row reduction methods to find the inverse A^{-1}

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_3-2R_1}{\overset{R_2-R_1}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 & 1 \end{bmatrix} \underset{-\frac{1}{2}R_3}{\overset{\frac{1}{2}R_2}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{-1}{2} \end{bmatrix}$$

$$\overset{R_1-R_3}{\longrightarrow} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{-1}{2} \end{bmatrix} \overset{R_1-R_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{-1}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

b) Use the inverse to solve the system of equations

$$\begin{aligned} x+y+z &= 0 \\ x+3y+z &= 2 \\ 2x+2y &= -4 \end{aligned}$$

Solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

c) Use the inverse and the properties of inverse and transpose matrices to solve the system of equations

$$\begin{aligned} x+y+2z &= 8 \\ x+3y+2z &= 12 \\ x+y &= 4 \end{aligned}$$

Solution: Write as a matrix: $B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix} = A^T$.

Moreover, $B^{-1} = \left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

5. Find the equation of the line going through (2,1,3) and parallel to the vector $\overrightarrow{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$

Solution:

$$\begin{aligned} x &= 3t+2 \\ y &= -2t+1 \\ z &= 4t+3 \end{aligned}$$

6. Suppose that $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. Prove, that $\{\overrightarrow{v_1}, \overrightarrow{v_1}+\overrightarrow{v_2}, \overrightarrow{v_1}+\overrightarrow{v_3}\}$ is a linearly independent set of vectors.
Solution:To be linearly independent, $c_1\overrightarrow{v_1}+c_2(\overrightarrow{v_1}+\overrightarrow{v_2})+c_3(\overrightarrow{v_1}+\overrightarrow{v_3}) = \boldsymbol{\theta}$ cannot have a nonzero solution. Equivalently, $(c_1+c_2+c_3)\overrightarrow{v_1}+c_2\overrightarrow{v_2}+c_3\overrightarrow{v_3} = \boldsymbol{\theta}$ cannot have a nonzero solution. But $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is a set of linearly independent vectors in $\mathbb{R}^n$ so the coefficients $(c_1+c_2+c_3) = 0, c_2 = 0, c_3 = 0$. Since both $c_2 = 0$ and $c_3 = 0, (c_1+0+0) = 0$, so $c_1 = 0$. Therefore $c_1 = c_2 = c_3 = 0$, and $\{\overrightarrow{v_1}, \overrightarrow{v_1}+\overrightarrow{v_2}, \overrightarrow{v_1}+\overrightarrow{v_3}\}$ is a linearly independent set of vectors.
7. Find the equation of the plane perpendicular to the line in problem 5 and going through the point (6,-3,5).

Solution: $\vec{v} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is normal to the plane. Therefore the equation of the plane is:

$$3x - 2y + 4z = D$$

Setting x=6, y=-3, z=5 yields $D = 3\,(6) - 2\,(-3) + 4(5) = 44 : 3x - 2y + 4z = 44$

8. For a three by three matrix, the elementary row operation of adding twice row 2 to three times row 3 and placing that in row 2 can be represented by a 3x3 matrix E. Write E.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Find the vector form for the general solution to the system of equations

$$\begin{aligned} x - 2y &= 0 \\ 2y + z &= 0 \\ 3x - 4y + z &= 0 \end{aligned}$$

Solution: Write as a matrix:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 3 & -4 & 1 \end{bmatrix} \overset{R_3 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \overset{R_3 - R_2}{\longrightarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let

$x - 2y = 0, 2y + z = 0 => x = 2y, y = \frac{-1}{2} z => x = -z =>$ the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$$