# MTH 362-01 TEST 2 Solutions
Fall 2008

1. Define / state
   a) a set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ of vectors is linearly independent
   b) a set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ of vectors spans a subspace $W$.
   c) a set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ of vectors is a basis for a subspace $W$
   d) a set $S$ is a null space of a matrix $A$
   e) $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.
   f) the dimension of a subspace $V$.
   g) the rank of a matrix $A$
   h) $\left\{\overrightarrow{b_1}, \overrightarrow{b_2}, ..., \overrightarrow{b_n}\right\}$ is an orthogonal basis for a subspace $W$
   i) $\left\{\overrightarrow{b_1}, \overrightarrow{b_2}, ..., \overrightarrow{b_n}\right\}$ is an orthonormal basis for a subspace $W$

   Soln: See text.

2. Find a basis for the subspace spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

   Soln: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$, so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$
is a basis.

3. Find a basis for the null space of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$

   Soln: From problem 2, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfies $x_1 = -\frac{2}{3}x_3,\ x_2 = \frac{1}{3}x_3$, so $\left\{ \begin{bmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$
is a basis

4. Let $A$ be an nxn matrix. Prove that $W = \{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} = \overrightarrow{x}\}$ is a subspace of $\mathbb{R}^n$.

   Soln: Let $W = \{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} = \overrightarrow{x}\}$ and $\overrightarrow{x}$ and $\overrightarrow{y}$ be elements of $W$. Then $A\overrightarrow{x} = \overrightarrow{x}$ and $A\overrightarrow{y} = \overrightarrow{y}$. Then $A(\overrightarrow{x} + \overrightarrow{y}) = A\overrightarrow{x} + A\overrightarrow{y} = \overrightarrow{x} + \overrightarrow{y}$, so $\overrightarrow{x} + \overrightarrow{y} \in W$. Let $c \in \mathbb{R}, \overrightarrow{x} \in W$, then $A(c\overrightarrow{x}) = cA\overrightarrow{x} = c\overrightarrow{x}$. Thus $W$ is a subspace of $\mathbb{R}^n$.

   Alternatively, $W$ is the nullspace of the matrix $A - I$, and hence a subspace.

5. Let $U$ and $V$ be subspaces of $\mathbb{R}^n$. Prove that $W = U \cap V$ is a subspace of $\mathbb{R}^n$.

   Soln: Let $U$ and $V$ be subspaces of $\mathbb{R}^n$ and $W = U \cap V$. Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be elements of $W$. Then $\overrightarrow{x}, \overrightarrow{y} \in U$ implies $\overrightarrow{x} + \overrightarrow{y} \in U$ and $\overrightarrow{x}, \overrightarrow{y} \in V$ implies $\overrightarrow{x} + \overrightarrow{y} \in V$ since $U$ and $V$ are subspaces. Therefore $\overrightarrow{x} + \overrightarrow{y} \in U \cap V$. Let $c \in \mathbb{R}, \overrightarrow{x} \in W$, then $\overrightarrow{x} \in U$ implies $c\overrightarrow{x} \in U$ since $U$ is a subspace and $\overrightarrow{x} \in V$ implies $c\overrightarrow{x} \in V$ since $V$ is a subspace. Therefore $c\overrightarrow{x} \in U \cap V = W$. Thus $W$ is a subspace of $\mathbb{R}^n$.

6. Find an orthonormal basis for the subspace spanned by the linearly independent vectors $\begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\6\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}$

Start with $\vec{v_1} = \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$. then $\vec{v_2} = \begin{bmatrix}3\\6\\2\\0\end{bmatrix} - \dfrac{\begin{bmatrix}0\\1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}3\\6\\2\\0\end{bmatrix}}{\left\| \begin{bmatrix}0\\1\\2\\0\end{bmatrix} \right\|^2} \begin{bmatrix}0\\1\\2\\0\end{bmatrix} = \begin{bmatrix}3\\6\\2\\0\end{bmatrix} - \frac{10}{5} \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$

$$= \begin{bmatrix}3\\6\\2\\0\end{bmatrix} - 2 \begin{bmatrix}0\\1\\2\\0\end{bmatrix} = \begin{bmatrix}3\\4\\-2\\0\end{bmatrix}$$

$$\vec{v_3} = \begin{bmatrix}2\\-1\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}}{\left\| \begin{bmatrix}0\\1\\2\\0\end{bmatrix} \right\|^2} \begin{bmatrix}0\\1\\2\\0\end{bmatrix} - \frac{\begin{bmatrix}3\\4\\-2\\0\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}}{\left\| \begin{bmatrix}3\\4\\-2\\0\end{bmatrix} \right\|^2} \begin{bmatrix}3\\4\\-2\\0\end{bmatrix}$$

$$= \begin{bmatrix}2\\-1\\1\\0\end{bmatrix} - \frac{1}{5} \begin{bmatrix}0\\1\\2\\0\end{bmatrix} - \frac{0}{29} \begin{bmatrix}3\\4\\-2\\0\end{bmatrix} = \begin{bmatrix}2\\-6/5\\3/5\\0\end{bmatrix}$$

An orthogonal basis is $\left\{ \begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\-2\\0\end{bmatrix}, \begin{bmatrix}2\\-6/5\\3/5\\0\end{bmatrix} \right\} = \left\{ \begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\-2\\0\end{bmatrix}, \begin{bmatrix}10\\-6\\3\\0\end{bmatrix} \right\}$.

An orthonormal basis is $\left\{ \begin{bmatrix}0\\1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}, \begin{bmatrix}3/\sqrt{29}\\4/\sqrt{29}\\-2/\sqrt{29}\\0\end{bmatrix}, \begin{bmatrix}10/\sqrt{145}\\-6/\sqrt{145}\\3/\sqrt{145}\\0\end{bmatrix} \right\}$

7. Find the matrix $A$ representing the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x-3y\\y-2z\\x+y+z\end{bmatrix}$.

the matrix is $A = \begin{bmatrix}2 & -3 & 0\\0 & 1 & -2\\1 & 1 & 1\end{bmatrix}$

8. Prove or disprove that $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 3/y \\ z \\ x - y + z \end{bmatrix}$ is a linear transformation.

Soln: T is not a linear transformation since $T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} undefined \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

It also violates both the scalar multiplication and additive properties.